Foundations for Success in College Mathematics

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Contents

Preface

Why write this book? That's a question that I needed to have a clear answer to before I began the process. The short answer is that I felt like I had little choice. Here is our reality:

- Nevada is historically among the lowest-ranked states for K-12 math education in the nation.
- The Nevada System of Higher Education passed a policy that completely eliminates any standalone math remediation for students, forcing us to adopt a full corequisite model.
- We want students to be successful in our college level math courses while maintaining our standards.
- The materials that currently exist are insufficient for the task.

And so by the necessity of the situation, we needed to come up with *something* to help our students.

But the longer answer to the question is that this moment of disruption is also the right time to try to make a change in how remedial mathematics (including corequisite courses) are taught and understood. I've spent over a decade working with students with weak mathematical backgrounds. I've watched how they think and how they attempt to learn mathematical material. I've come to the conclusion that I think there are better ways to help students become successful.

Of course, if we are talking about "success" then we need to define what that actually means. I will say that the traditional methods of remediation are pretty good at doing what they are designed to do. The primary emphasis of traditional remediation is to help students become proficient at specific algebraic manipulations. The entire structure of textbooks is based on that premise. Here is an example. Now try a problem exactly like that one. Now do that 50 more times (but only the odd numbered problems so you can check your answer). If all you want out of students is to get them to perform manipulations on command, this is a very good way of doing it.

But as I think about the students I encounter, I don't think this is really that helpful. I question whether they actually remain proficient in those manipulations after the semester is over. After all, they've been through this before, perhaps even two or three times. Why is this time going to be different? I also question what they actually learned from the class, and I question the true value that the students get from courses like these. It all comes back down to a question that's at the core of education: What do we *really* want students to learn?

If we only have one semester to teach students about mathematics, do we really want to teach them that the core of mathematics is doing algebraic manipulations? Is that what math is? Is that what we actually care about? I understand that some people would agree that it is. They think the goal is to get students to be proficient at these particular algebraic manipulations so that they can execute those manipulations in their other courses (college algebra/precalculus, physics, chemistry, statistics). I don't think this is wrong, I just think it's short-sighted.

I believe that the core skill that students should get is not mathematical manipulations, but the development of mathematical reasoning. The emphasis of the K-12 system is still currently heavily invested in mathematical manipulations. If you look at most Algebra 2 textbooks, you'll

Here is a subset of topics that may appear in a high school Algebra 2 course: rationalizing the denominator of radical expressions, complex numbers and the fundamental theorem of algebra, exponentials, logarithms, matrices, synthetic division, logic, probability, infinite series...

find an incredibly broad range of topics that are covered. Most of these topics end up being manipulations piled on top of other manipulations.

When I look at students in remedial courses, I make two primary observations about them. The first is that students are often very confused about mathematics. They operate from a very rule-based perspective and often feel as though the bulk of their work is memorizing manipulations and memorizing when they need to use them. The second, which follows from the first, is that they completely lack confidence in their mathematical abilities. This is a learned helplessness from all the times they tried to memorize something and failed. They have had many years to develop the "not a math person" identity where they do not embody any level of mathematical confidence and show few signs of mathematical reasoning. Many students simply guess at whether they are doing the right thing, then sit back and wait to be told whether it's right or wrong. And when you look at the educational system that they've come through, you can understand why this is.

My goal with this book is to change how students think about mathematics. I would love to have two or three semesters of math courses to really bring students to a place of thinking about mathematics at a college level. But there is only so much that can be done in one semester, especially when that semester is contextualized as a support for a college level math course. And that's the reality. We simply need to do the best we can with the time that we have.

Who is the Target Audience?

This book is written for all college students who are interested in improving their mathematical foundation. This includes students that are either destined for a liberal arts or statistics college math course, and students on the STEM trajectory headed towards calculus. I also think this has value to anyone who is teaching or will teach math at any level (especially K-12). There is a lot of systemic brokenness in the math education system, and I will be glad if these ideas help to fix even a smallest part of that.

The Goal of the Book

So we return to the driving question: What do we *really* want students to learn? The answer that this book gives is that the goal is to get students to become better and more confident mathematical thinkers. They've spent enough time doing the lather-rinse-repeat of pure manipulations. They need to develop a different intellectual foundation. The good news is that we're not working from scratch. Because although students may struggle with certain types of algebraic manipulations, it's rare that all the students need to relearn all of them.

So instead of treating students as if they've never seen these manipulations before (which is how the majority of remedial coursework approaches topics), we're going to work with students as people who have experience but have not carefully reflected on those experiences. A lot of the manipulations that we ask of them are already somewhere in their heads, and we're simply working with them to connect those neurons to other neurons and strengthen that signal.

This begins with the very first section. The emphasis is on mathematical communication. This is already a significant divergence from most other approaches, where the goal is to get the right answer. The majority of students don't even have a basic framework for understanding

Instructors will likely be pleased by students having an organized framework for writing their mathematics, even if the students are doing everything wrong. At least it's legible!

mathematical communication. We lay out simple but clear expectations for how to begin to organize their mathematical writing, and that foundation is the tool that we use to help students reorganize the information in their heads. Once we can get them to write their work in an organized manner, it becomes easier to begin to isolate specific struggles that students are having, and they can even begin to start to recognize them for themselves.

Every section that follows treats students as adults who already have mathematical experiences. We do not treat the students as kids who need to have to be told exactly what to do all the time. In fact, we emphasize throughout the book that mathematics is not about following rules, but about being able to think through situations and understanding what they're doing and why they're doing it. You will find that some of the "Try It" problems do not have an example that models the exact thing they need to do. Those examples often become a crutch for students as they realize that they don't actually need to think for themselves, and simply have to hunt down the right example to show them exactly what to do. You will see this philosophy expressed in the worksheets as well, as many of them will touch topics that were not directly discussed in the text.

We have done this specifically because we want students to learn to think for themselves. The goal is not just that students will learn the idea, but that they will begin to develop metacognitive strategies for how they approach new mathematical ideas. This is a much broader foundation that they are more likely to carry forward with them into their future classes. If you just teach the manipulations, then it's going to be hit-or-miss whether they will remember those manipulations when they need them in the future. But if you give them the tools to think about the mathematics effectively (and the confidence to trust their thinking), they will be far more able to reconstruct the ideas behind the manipulations if they've forgotten them. It simply puts them in a much better position for long-term success.

The Structure of the Book

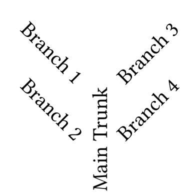
One of the features of this book to help students accomplish this is that only part of the book is intended to be "taught" in the classroom. Self-efficacy comes with the practice of being self-efficacious. This was an intentional decision based on the use of this material as part of a corequisite math course. As much as we can want to aim for deeper learning outcomes, we still need to confront the practical reality of our students' struggles with mathematics.

In fact, the whole book can be used as a self-study book if one is independently motivated. But most students will need to transition into it.

The book is broken into five sections:

- The Main Trunk: Core algebra (Sections 1-10)
- Branch 1: Linear equations and the coordinate plane (Sections 11-15)
- Branch 2: Fractions and decimals (Sections 16-21)
- Branch 3: A review of arithmetic (Sections 22-29)
- Branch 4: A few applications (Sections 30-32)

We view the content like a tree. After covering the main trunk, the branches can be done as distinct topics.



The main trunk is the set of core algebra that we think are absolutely critical before launching into any college level mathematics. We spend the first two weeks of the semester covering these sections. They are meant to be reminders of things students have already learned, not a complete reteaching of the content from scratch. The most important section is the first one, where we introduce the idea of mathematical communication. In that section, we provide students a framework that we expect them to follow. The idea is to slow them down and get them to think about the algebraic manipulations, and to otherwise disrupt the bad habits they've developed. Without this shift, it's significantly more difficult for students to make the necessary changes.

The presentation framework is to line up the equal signs vertically and use the space to the right to explain or justify the steps. And we really do make students do this!

Sections 8 and 9 (factoring quadratic polynomials) can be skipped without loss of continuity if these skills are deemed unnecessary for the students.

Beyond the main trunk, the students are expected to work on the sections more or less on their own. We do have times of in-class activities that will include demonstrating that they've worked through these sections, but we generally do not intend for our instructors to directly teach out of the book after completing the main trunk. The textbook is written in a conversational tone that students should be able to read and understand. Rather than being a few big, complex ideas, it's really more a collection of a lot of little ideas to help math "make sense" to students. The instructors are expected to be available to help students if they get stuck on topics, but we are generally confident in the students' ability to mostly work things out on their own.

Yes, we also expect students to *read* the words in the textbook!

We win when we get this reaction from students: "Oh! Now I understand why we do it this way!"

The emphasis of the branches is not about getting students to execute technical manipulations. It's about getting them to start to think and interact differently with mathematical concepts. Remember that the goal is to have increased mathematical thinking and mathematical confidence, not growing their catalog of algebraic manipulations. So while we do also review manipulations (such as fraction and decimal calculations), the context and presentation lend themselves more towards thinking accurately about these mathematical ideas and not treating them as a series of rule-based manipulations.

This perspective is the opposite of how many math textbooks are written. Most of the time, there is a core thread of content and the branches point outward to bigger ideas and more distant horizons. For this book, the branches point inward to the main trunk. Rather than using the main trunk as a launching point for new ideas, it is the home base that we return to over and over again. And this is how we achieve the depth of mathematical thinking.

We would rather that students make the mental effort to connect the dots of ideas that already exist in their heads instead of asking them to pursue new topics.

The college level math course is where students will face new ideas!

The Structure of Each Section

The core structure of each section is the following:

- Learning Objectives: Each section has 1 to 4 learning objectives. This gives a brief summary of the core concepts in the section.
- Main Content with Examples: Students are expected to read the text and complete the "Try It" examples. If students are reading the text, they should be able to do these problems. And if they can't do these problems, that's a signal to them that they need to contact the instructor (or campus tutoring services, if available) to get help. The solutions to these examples are provided at the end of each section.
- Worksheets: The worksheets are extra problems that are sometimes similar to the "Try It" examples, but they also sometimes diverge into a deeper look at familiar ideas. Those prob-

lems are where students are being asked to think mathematically. Given what they know, can they push deeper and make a new connection? We encourage students to approach these as group explorations because the act of talking through ideas (communication) is important to the primary learning outcomes. The worksheets are intentionally short to allow for these types of conversations to happen relatively quickly, rather than having them first do dozens of rote exercises before sharing their ideas with each other.

- Deliberate Practice: Most sections will have a short collection of exercises to allow students a bit of additional practice. This is called "deliberate practice" because students are given a number of ideas to focus on while they do these problems. These are supplements to provide students with further opportunities to practice the skills that they've developed (thinking, presenting, explaining, and executing) by doing standard manipulations. If you are looking to just dump a bunch of problems on students, you're much better off using a different textbook. These sections are intentionally kept short to discourage instructors from just assigning students to do rote exercises.

- Closing Ideas: Each section has a brief discussion that summarizes the ideas that are found in the section and the worksheets. The purpose of this is to redirect the focus towards the horizon. As students work their way through each section, it's easy to get fully absorbed in executing the calculations and lose track of how this fits into the bigger picture.

- Going Deeper: Many sections will have an additional topic that pushes students deeper. These sections exist to better illuminate the ways that the ideas from the section link to higher levels of mathematics and other applications of the ideas in each section. These additional topics can all be skipped without losing any of the core content of the book, but some instructors may find some of the topics to be helpful for use in just-in-time remediation. For example, there is a series of sections that focus on manipulating rational expressions that a precalculus course may want to cover. Other sections are there simply as fodder for discussion and to broaden students' mathematical perspective.

- Solutions: Solutions to the "Try It" examples have been provided. It is important to emphasize that these solutions are presented using the presentation expectations that are asked of students to emphasize the point that getting the right answer isn't the only goal.

Alignment with Precalculus Courses

This textbook was organized with precalculus in mind. While precalculus textbooks vary somewhat, they mostly fit the same basic pattern. This means that this textbook should work well with most standard first semester precalculus courses. We wanted to create a rough alignment between this material and the basic structure those courses. Specifically, there were three major touch points that we wanted to have:

- Discuss the coordinate plane and graphing lines around the same time students are studying the general properties of functions.

- Discuss of fractions before introducing of rational functions to remind students of the underlying manipulations before they need it.

- Discuss scientific notation around the time as logarithmic functions to create the opportunity to connect the two ideas.

The suggested course alignment below is based on having extra contact time with students for the corequisite content, which allows for the extra time up front to set the stage with the Main Branch of this book. (I don't know how you would try to teach both the support material and the core content without extra contact time!) It's possible to use this material without it, but students may not adopt the mindset and writing habits if those sections are not treated as the primary learning outcomes. After the Main Branch, very little in-class time is spent directly teaching the Foundations content. Students are expected to read the book, and only a small portion of the class time is dedicated to completing the worksheets and discussing the ideas directly.

	Foundations Textbook	Precalculus Content
Weeks 1-2	Main Branch (Core Algebra)	
Weeks 3-5	Branch 1 (Linear Equations/Coordinate Plane)	Introduction to Functions
Weeks 6-10	Branch 2 (Fractions/Decimals)	Polynomial and Rational Functions
Weeks 11-15	Branches 3-4 (Arithmetic/Applications)	Exponential and Logarithmic Functions

Acknowledgments and Closing Comments

These materials would not be possible without the students at Nevada State College, especially those who have shared their math struggles with me. Without hearing your experiences and perspectives, I may not have developed the outlook that has ultimately led to the framework used in this textbook. I'm sorry for the ways that the beauty of math has been obscured by the educational system, and I hope that some day you'll be able to appreciate it for what it really is.

Additionally, I'd like to thank Seth Churchman for his encouragement, his feedback, and his other contributions to these materials.

How We Write Affects How We Think: Basic Algebraic Presentation

Learning Objectives:

- ☐ Present a sequence of algebraic manipulations with the equal signs lined up.
- ☐ State an algebraic manipulation in words and then execute that manipulation correctly.

Welcome to college level mathematics. It is a well-known fact that many students struggle with college mathematics. There are many people who proudly announce that they took a college math course three or four times before they passed. But what is less known are the reasons why students struggle with college level mathematics. Some will cite math anxiety, others may blame the instructor, and others will simply say that they're not math people. And there's probably a hint of truth to all of those.

What does it say about the state of math education when people are proud that they failed math classes?

Growth Mindset: "I'm not a math person ... yet."

But a more basic reason for struggling with mathematics is that most students simply have not learned to think mathematically before they get to college. Maybe they've memorized some formulas and some basic ways to manipulate equations, but there's often a gap of understanding and the inability to communicate mathematics effectively. The easiest way to spot this is when a student says, "I know what I'm doing, but I can't explain it." This might be good enough at lower levels of mathematics, where the goal is simply to get the answer, but college level mathematics is different. We want students to be able to explain what they're doing and why.

Have you ever said that?

These materials are designed to help students bridge that gap so that they can become mathematical thinkers. But what is mathematical thinking? While it's hard to describe it in complete detail, it includes (and is not limited to) the following:

- The ability to think logically and analytically to solve problems
- The proper manipulation of symbols in the process of solving problems
- The ability to communicate and explain the ideas behind algebraic manipulations and the reasons for using them

A couple more:

- The confidence to attempt to solve problems using mathematics.
- The curiosity to ask questions about mathematical ideas.

As you work your way through these materials, you should be finding yourself being able to do these things better through thoughtful practice and repetition. Just as with a foreign language, it's not enough to just say a phrase once to learn it. You need to do it over and over again, and you need to see it or hear in multiple contexts.

You will often hear people say that math is a language. It can be useful to think of it in that way.

1 We're going to start with some basic ideas for communicating mathematics. For the purposes of this book, a complete presentation of a mathematical manipulation includes the following pieces:

- A series of equations or expressions with the equal signs lined up vertically
- An explanation of the manipulations on the right side of each manipulation

These are not "rules" that must be followed all the time. But they are good guidelines for what clean mathematical presentation should look like.

Here is an example of solving the equation $3x + 4 = 16$ using a complete presentation:

$$\begin{aligned} 3x + 4 &= 16 & & \\ 3x &= 12 & & \text{Subtract 4 from both sides} \\ x &= 4 & & \text{Divide both sides by 3} \end{aligned}$$

Try it: Using the above example as a model, solve the equation $5x - 7 = 23$.

Most textbooks use this format when presenting examples.

2 Lining up the equal signs is mostly a matter of organization and readability. Some students learned to do this because then they can use up the entire width of their paper and can cram in more problems per page. Unfortunately, as problems become more complex, this leads to all sorts of small errors that could easily be avoided by a more organized presentation.

Consider the following set of equations:

$$5x + 9 = -3x - 7 \qquad 5x = -3x + 16 \qquad 2x = 16 \qquad x = 8$$

There are two errors embedded into these calculations. Notice how much your eyes have to go back and forth to match up the terms. Lining up the equal signs vertically reduce that distance and make it easier to see when terms change or disappear. It also helps you to keep track of what is on the left side of the equation and what is on the right side of the equation.

Try it: Present the calculation above using a complete presentation. Be sure to fix the errors.

If you want to be environmentally conscious, you can work in two or three columns on your page instead of leaving the right half of the paper blank.

This exercise is much more difficult with handwritten math. Typed math is so clean that the errors easier to find.

If you aren't familiar with solving equations like this, talk to your instructor for further help.

3 There is a common way of writing equations that some people use which is less than ideal:

$$\begin{array}{rcr} 4x + 7 & = & 19 \\ -7 & & -7 \\ \hline \frac{4x}{4} & = & \frac{12}{4} \\ x & = & 3 \end{array}$$

There are a number of problems with this. To start, there's not really an organized way to read this. The best way to think of it is that it breaks apart into four pieces with three of them overlapping each other. Another problem is that the reader is expected to simply know what's happening. This is sometimes at the root of students' complaints about "skipped steps."

Original Problem	Subtract 7 to get a new equation	Divide the new equation by 4	Final answer
$4x + 7 = 19$	$\begin{array}{rcr} 4x + 7 & = & 19 \\ -7 & & -7 \\ \hline 4x & = & 12 \end{array}$	$\frac{4x}{4} = \frac{12}{4}$	$x = 3$

Try it: Fix the above calculation and write it as a complete presentation.

You probably had a teacher teach you that this is the right way to write math. Unfortunately, they were wrong and you're going to need to unlearn it.

A good explanation of an idea should not require the reader to have to guess about what happened. This means that the responsibility is on the writer to write clearly.

This all makes sense as you're writing it out, and it's good to help you put your ideas together and get the overall picture of what's happening in a problem, but the final result isn't a carefully organized presentation. In writing courses, the equivalent work would be called a "rough draft."

4 One other aspect of taking the time to carefully write what you're doing is that it causes you to think more clearly about what you're doing. Consider the following equation: $2x = \frac{1}{2}$. A fair number of students see this and think they should "cancel out" the 2 from both sides. This is not correct. There are times you can cancel out numbers if one of them is in the numerator and the other is in the denominator, but this isn't one of them.

The act of explicitly stating what mathematical operation you're performing helps your brain to make categories of information. A more simple "error" of this type is that some students refer to all algebraic manipulations as "moving the term to the other side." Here are two examples of what that could mean to students:

$$\begin{aligned} 3x + 5 &= 14 \\ 3x &= 9 \end{aligned} \qquad\qquad \begin{aligned} 3x - 7 &= 14 \\ 3x &= 21 \end{aligned}$$

Each of these is a different mathematical operation and a different algebraic step. It is not uncommon to see the following mistakes:

$$\begin{aligned} 3x + 5 &= 14 \\ 3x &\overset{\times}{=} 19 \end{aligned} \qquad\qquad \begin{aligned} 3x - 7 &= 14 \\ 3x &\overset{\times}{=} 7 \end{aligned}$$

Usually, when I ask students to state in words what they did, they're able to see their mistake for themselves. So the act of stating the mathematical operation in words allows students to avoid errors.

Try it: Solve the equations $3x + 5 = 14$ and $3x - 7 = 14$ using a complete presentation.

Working with fractions is its own challenge for a lot of students. A lot of students have very little conceptual foundation for the fraction manipulations, and a lot of it is just brute force memorization. We'll try to enlighten fractions a bit later so that they really start to make sense.

We use $\overset{\times}{=}$ to indicate that there was an algebraic error.

Metacognition is the awareness of your own thinking. Many students lack this with mathematics, and are simply manipulating symbols on the page. By thinking about the algebraic steps you're executing, you give yourself better learning opportunities.

You will not always need to use a complete presentation. There are many times when it is tedious and unnecessary to state what is happening at every single algebraic step. This becomes more and more true as your basic algebra gets stronger and stronger. What ends up happening is that the steps that need to be described become less about the small details and more about the big picture. When solving for x, it may be enough to say, "Apply the quadratic formula" and not have to explain all of the individual arithmetic steps.

That being said, it is very important to develop the habit of keeping your equal signs lined up when writing math. If there is any one writing habit you develop from these worksheets, it should be that one. This one change by itself does a lot of things for you that you may not be consciously aware of.

1.1 Basic Algebraic Presentation - Worksheet 1

1 Circle and identify the two features that make the equations below make a complete presentation of the mathematical manipulation.

$-6x + 16 = -8$	
$-6x = -24$	Subtract 16 from both sides
$x = 4$	Divide both sides by 6

2 Solve the equation $5x + 9 = 34$ using a complete presentation.

3 Solve the equation $4x - 13 = 8$ using a complete presentation.

Don't assume that if you end up with a fraction that you must have done something wrong! You will need to develop the self-confidence to assess your work based on the logic and not the answer.

4 Work backwards from the given information to derive the original presentation.

	Add 8 to both sides
$x = -3$	Divide both sides by 5

Problem solving is an important life skill. If you can only work with information when it's presented in one specific way, you don't have a complete understanding of it.

1.2 Basic Algebraic Presentation - Worksheet 2

1 Solve the equation $6x + 9 = -21$ using a complete presentation.

2 Solve the equation $4x + 7 = 7$ using a complete presentation.

Students sometimes get confused when division involves the number zero. We'll look into division in detail later. For now, just remember that zero divided by anything is zero and that you should never divide by zero.

3 Solve the equation $-5t - 27 = -6t$ using a complete presentation.

While we traditionally use the variable x, we can and will use many other symbols as our variables.

4 Work backwards from the given information to derive the original presentation.

	Add $3y$ to both sides
	Subtract 6 from both sides
$7 = y$	Divide both sides by -2
$y = 7$	Rewrite in the conventional order

Sometimes, in the process of solving an equation, you may end up with the variable on the right side. It is conventional to have your final answer written with the variable on the left side, even if both $7 = y$ and $y = 7$ mean the same thing. It's not a big deal if you don't do this, but it's a nice thing to do.

1.3 Basic Algebraic Presentation - Worksheet 3

1 Solve the equation $-9x+3=66$ using a complete presentation.

2 Solve the equation $3x+9=-9$ using a complete presentation.

If you get $x=0$, then you made an error. Be careful because this is a common mistake!

3 Solve the equation $-8x+17=-4x+41$ using a complete presentation.

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$-9x+16=7x-8$	
$-9x=7x-24$	Subtract 16 from both sides
$-2x=-24$	Subtract $7x$ from both sides
$x=12$	Divide both sides by -2

It is usually much easier for us to see errors in other people's work than it is to see them in our own. One way to help find your own errors is to walk away for a while and come back later. This gives your brain a chance to "forget" what you did so that you can rethink everything from the beginning. Wait... did you think this is a discussion about algebra?

1.4 Basic Algebraic Presentation - Worksheet 4

1 Solve the equation $-3x + 12 = 5x - 8$ using a complete presentation.

2 Solve the equation $4x + 7 = -3x + 7$ using a complete presentation.

3 Perform the indicated algebraic manipulations.

This is known as "completing the square." You may not know what it is or why you're doing it right now, but this is an important algebraic concept that will likely see later. For now, just focus on practicing your algebra.

$$x^2 = 6x + 7$$

Subtract $6x$ from both sides

Add 9 to both sides

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$5x + 8 = -2x - 8$	
$5x = -2x$	Subtract 8 from both sides
$7x = 0$	Add $2x$ to both sides
$x = 0$	Divide both sides by 7

1.5 Basic Algebraic Presentation - Worksheet 5

1 Solve the equation $7x - 15 = -2x + 7$ using a complete presentation.

Don't assume you've made a mistake just because you got a fraction.

2 Solve the equation $-5x - 3 = -3x + 3$ using a complete presentation.

3 Work backwards from the given information to derive the original presentation.

There is no rhyme or reason to the manipulations for this one. It's just an exercise in thinking carefully.

	Subtract $3x$ from both sides
$2x + 1 = 5x - 4$	Add 9 to both sides

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$6x + 4 = -3x - 10$	
$6x = -3x - 14$	Subtract 4 from both sides
$9x = -14$	Add $3x$ to both sides
$x = \frac{14}{9}$	Divide both sides by 9

1.6 Deliberate Practice: Solving Equations for a Variable (Part 1)

Algebra is a skill, which means it requires practice to become proficient. But it will take more than rote repetition to get there. Deliberate practice is the thoughtful repetition of a task. For each of these sections, you will be given a list of specific skills or ideas to focus on as you practice thinking through the problems.

Focus on these skills:

- Write the original equation.
- Line up your equal signs.
- Correctly state the algebraic steps using the correct phrasing:
  - Add (expression) to both sides.
  - Subtract (expression) from both sides.
  - Multiply both sides by (expression).
  - Divide both sides by (expression).
- Execute the algebra and arithmetic correctly.
- Present your work legibly.

Instructions: Solve the equations for the variable using a complete presentation.

1. $7x + 34 = 13$
2. $2y + 24 = -32$
3. $5t - 7 = 7$
4. $6u - 18 = -31$
5. $3a - 44 = -38$
6. $2z + 24 = 24$
7. $5b - 1 = 40$
8. $11x + 40 = 13$
9. $6t - 19 = -35$
10. $9y - 40 = 32$
11. $-4u - 31 = -25u - 10$
12. $4v - 40 = -14v + 14$
13. $7r + 38 = 8r + 32$
14. $-9b + 3 = -8b + 14$
15. $6c + 7 = 7c + 13$
16. $2t - 10 = -3t + 10$
17. $17u - 5 = 3u + 23$
18. $-x - 1 = 2x + 83$
19. $2m + 6 = 4m - 2$
20. $s + 9 = -2s + 18$

1.7 Closing Ideas

We have been implicitly using some very core mathematical ideas throughout this section. Specifically, we have been working with the *axioms of equality*. All of the axioms are built on the idea that if you start with two quantities that are equal, and then perform the same manipulation to both of them, that the results should be equal. Here's what this formally looks like:

A mathematical axiom is a pattern that we accept to be true without a formal mathematical proof.

Definition 1.1. Let a, b, and c be real numbers. The *axioms of equality* state that

- If $a = b$, then $a + c = b + c$.
- If $a = b$, then $a - c = b - c$.
- If $a = b$, then $ac = bc$.
- If $a = b$, then $\frac{a}{c} = \frac{b}{c}$, provided that $c \neq 0$.

There are actually even more axioms of equality! But here we want to focus on the algebraic manipulations.

This may look intimidating at first, but it's much less so once you recognize that this is what you've been doing the entire section. It just says that if we add to, subtract from, multiply, or divide (as long as we don't divide by zero) both sides by the same value, then we maintain the equality. And this is what allows all of our algebraic manipulations to make sense.

Why can't we divide by zero?

It should also be fairly intuitive that if we started with an equality but treated the two sides differently, we would break the equality. For example, if we start off with $5 = 5$, but then add 3 to the left side and subtract 2 from the right side, the two sides will simply be different values.

The important takeaway from this is that these axioms simply make sense. It's hard to imagine a world where we each start with five apples, we both eat one of our own apples, but we somehow end up with two different numbers of apples. And that's supposed to be true about math in general. Math is supposed to make sense. In some ways, mathematics represents the epitome of strict logical reasoning.

Unfortunately, for many students, math is anything but that. There are so many symbols, so many rules, and so many manipulations that they have been forced to memorize that the whole thing is an incomprehensible mess. This book is our attempt to change that. The focus here is not going to be just on manipulating symbols, but really understanding what's going on. The emphasis will be placed on making sure you are clear in your thinking and clear in your communication. These things are far more important than just running you through the gamut of algebraic manipulations for the second, third, or fourth time in your life.

The hope is that as you start to think about math differently, you will find more and more of it simply makes sense.

1.8 Going Deeper: Clearing the Denominator

For the problems in this section, it was possible that you ended up with a fraction in your final answer, but you were not given any fractions in the original problems. Students tend to have a negative relationship with fractions. Later in the book, we're going to take a very close look at them. But for now, we're going to focus on just some basic algebra involving fractions.

Many students are at least vaguely familiar with something called "cross multiplication." This is a technique for working with the specific situation when you have two fractions equal to each other and you're trying to solve for the variable. Here is how it's often portrayed:

Notice how if you don't really know what's going on, the presentation is extremely unhelpful. It looks like you've just crossed out the whole problem.

$$\frac{a}{b} = \frac{c}{d}$$
$$ad = bc$$

Many teachers teach this method, but there are some downsides to it. It creates an additional rule for students to remember, it's often applied incorrectly, and it doesn't teach or reinforce any specific algebraic concepts.

Fractions in equations introduce their own set of difficulties because fraction manipulations involve their own concepts (for example, common denominators and reducing) which make things more complicated. Unfortunately, this leads to the tendency to create even more manipulations for students to learn so that you have one set of rules for fraction equations and a different set of rules for non-fraction equations. (If you're not comfortable with fractions, there's a very brief fraction review at the end of Section 15, and a more thorough discussion that runs through Sections 17, 18, 19, and 20.)

If you look back at Definition 1.1, you might notice that there's one algebraic step that we did not use in this section. We never multiplied both sides of the equation by a value in our attempts to solve for the variable.

Go back and check this claim!

A very broad category of algebraic manipulations is known as "clearing the denominators." And this technique is built on multiplying both sides of the equation by the same value. The basic premise of this method is that it's often much easier to solve equations when there aren't fractions, and so it can make sense to eliminate them from the equation as the first step.

$$\begin{aligned} \frac{3x}{4} &= \frac{5}{3} && \\ 12 \cdot \frac{3x}{4} &= 12 \cdot \frac{5}{3} && \text{Multiply both sides by 12} \\ 3 \cdot \cancel{4} \cdot \frac{3x}{\cancel{4}} &= \cancel{3} \cdot 4 \cdot \frac{5}{\cancel{3}} && \text{Reduce} \\ 9x &= 20 && \text{Simplify} \end{aligned}$$

You can argue that cross-multiplication is "faster" in this case, and you would probably be correct. But that doesn't mean that cross-multiplication isn't without its drawbacks. The blind application of cross-multiplication can lead to numbers that are larger than necessary. Consider the following:

$$\begin{aligned} \frac{5x}{12} &= \frac{3}{8} \\ 24 \cdot \frac{5x}{12} &= 24 \cdot \frac{3}{8} && \text{Multiply both sides by 24} \\ 2 \cdot \cancel{12} \cdot \frac{5x}{\cancel{12}} &= 3 \cdot \cancel{8} \cdot \frac{3}{\cancel{8}} && \text{Reduce} \\ 10x &= 9 && \text{Simplify} \end{aligned}$$

If we were to use cross-multiplication, we would have ended up with $40x = 36$. Then after dividing, we would have had to reduce the fraction, which creates more opportunities for error.

We can do even better by multiplying both sides by $\frac{12}{5}$, but that requires a higher level of skill and confidence with fraction manipulations.

And then there's the challenge of more complex equations. Consider the following:

$$\frac{3x}{4} + \frac{1}{3} = \frac{5}{6}$$

Students that have learned cross-multiplication will often try to apply it to this situation, even though it doesn't apply. When students learn rule-based mathematics, they will go through all sorts of interesting machinations to try to apply a rule where it doesn't belong because they simply don't know what else to do.

"If the only tool you have is a hammer, then everything looks like a nail."

We could solve this using fraction manipulations, but most people (including instructors) prefer not to do that if they don't have to. And clearing the denominator is the way around that. The trick is to multiply both sides of the equation by a number that causes all the denominators to cancel out.

This value turns out to be the *least common denominator* of all the fractions.

$$\begin{aligned} \frac{3x}{4} + \frac{1}{3} &= \frac{5}{6} \\ 12 \cdot \left(\frac{3x}{4} + \frac{1}{3} \right) &= 12 \cdot \frac{5}{6} && \text{Multiply both sides by 12} \\ 12 \cdot \frac{3x}{4} + 12 \cdot \frac{1}{3} &= 12 \cdot \frac{5}{6} && \text{Distribute} \\ 3 \cdot \cancel{4} \cdot \frac{3x}{\cancel{4}} + \cancel{3} \cdot 4 \cdot \frac{1}{\cancel{3}} &= 2 \cdot \cancel{6} \cdot \frac{5}{\cancel{6}} && \text{Reduce} \\ 9x + 4 &= 10 && \text{Simplify} \end{aligned}$$

Notice the parentheses! Those are incredibly important here!

If you don't remember the distributive property, it's coming up in a couple sections.

This technique is used much further along in mathematics. For example, there's a technique called "partial fraction decomposition" that is used in calculus to break apart a fraction into simpler components. It comes down to working with an equation that might look like the following:

$$\frac{A}{x-2} + \frac{B}{x+1} = \frac{x+4}{(x-2)(x+1)}$$

The goal is to solve for A and B. A common first step is to clear the denominators so that you don't have to deal with fractions, and that manipulation looks very similar to the one above.

$$\begin{aligned}
\frac{A}{x-2} + \frac{B}{x+1} &= \frac{x+4}{(x-2)(x+1)} \\
(x-2)(x+1) \cdot \left(\frac{A}{x-2} + \frac{B}{x+1} \right) &= (x-2)(x+1) \cdot \frac{x+4}{(x-2)(x+1)} \\
(x-2)(x+1) \cdot \frac{A}{x-2} + (x-2)(x+1) \cdot \frac{B}{x+1} &= (x-2)(x+1) \cdot \frac{x+4}{(x-2)(x+1)} \\
\cancel{(x-2)}(x+1) \cdot \frac{A}{\cancel{x-2}} + (x-2)\cancel{(x+1)} \cdot \frac{B}{\cancel{x+1}} &= \cancel{(x-2)}\cancel{(x+1)} \cdot \frac{x+4}{\cancel{(x-2)}\cancel{(x+1)}} \\
(x+1)A + (x-2)B &= x+4
\end{aligned}$$

At that level, steps like this are sufficiently routine that we usually don't write out the description of every single step. But that doesn't mean that we aren't thinking the words.

Can you match up the steps in this calculation with the one above?

This is only the first step of this part of the problem. In practice, the next step would be to determine the values of A and B, then use this equation to substitute for the integrand of an integral, which then needs to be integrated.

You're not expected to know what all those words mean.

Fortunately, you don't need to worry about this right now. The point is that the algebraic techniques you're learning right now are the same algebraic techniques you're going to see down the line, especially if you are on a track that's taking you towards calculus. It is important to do your best to build a solid foundation now so that you will be ready when you see more complicated manipulations in the future. Because it is an incredibly difficult task to do both at the same time.

1.9 Solutions to the "Try It" Examples

1

$5x - 7 = 23$	
$5x = 30$	Add 7 to both sides
$x = 6$	Divide both sides by 5

2

$5x + 9 = -3x - 7$	
$5x = -3x - 16$	Subtract 9 from both sides
$8x = -16$	Add $3x$ to both sides
$x = -2$	Divide both sides by 8

The two errors:

- Line 2: Wrong sign on the 16
- Line 3: Calculation error

3

$4x + 7 = 19$	
$4x = 12$	Subtract 7 from both sides
$x = 3$	Divide both sides by 4

4

$3x + 5 = 14$	
$3x = 9$	Subtract 5 from both sides
$x = 3$	Divide both sides by 3

$3x - 7 = 14$	
$3x = 21$	Add 7 to both sides
$x = 7$	Divide both sides by 3

Letters are Numbers in Disguise: Variables in Expressions and Equations

Learning Objectives:

- ☐ Substitute a number for a variable and then simplify the expression.
- ☐ Manipulate linear equations with multiple variables.

Some people like to say that math made sense when it was all numbers, but then things fell apart when the letters started showing up. This is unfortunate, because the letters are actually just a way of representing numbers, and thinking about them in that way can help to clarify confusion. There are a lot of ways that teachers talk about variables and they are of varying degrees of correctness. For example, some teachers say that variables represent unknown values. It's not exactly wrong, but it's not exactly right. Whether or not we know the value is irrelevant. In college level math courses, we can use variables to represent mathematical expressions with other variables in it instead of just thinking of it as a quantity. This leads us to a more general definition:

In broader contexts, variables are viewed in different ways. In computer programming, variables are containers for information. In higher level math courses, variables are sometimes used for abstract mathematical objects. The way we are using it here is consistent with both perspectives.

Definition 2.1. A *variable* is a symbol that represents a quantity or a mathematical expression.

Mathematicians love to have good definitions. They help us to think about our ideas with very high levels of precision.

Notice that we allow for variables to represent both quantities and mathematical expressions. What this means is that both $x = 4$ and $x = y + 1$ are valid uses of variables.

1 What does it mean for a variable to represent a quantity in an expression? One way to think about it is that it's a calculation waiting to happen. Consider the expression $2x$. This represents a number, but which number it represents depends on the value of x. As soon as I give you a value for x, you can plug it in and give me a specific number. But until then, this is just a number that is waiting to be calculated. For example, if you know that $x = 4$, then you can calculate that $2x = 8$. We can write this as an English sentence:

If $x = 4$, then $2x = 8$.

Notice that we're not saying that $2x$ always has the value of 8. We're just saying that *if* x takes the value of 4, *then* the value of $2x$ must be 8.

Try it: Determine the value of $2x$ when $x = 5$, $x = 12$, and $x = -3$. Write your results as if-then statements.

It turns out that the if-then construction is at the core of basically our entire process of logical reasoning. You will see it in a wide range of courses, from an introduction to philosophy to an introduction to computer science. It's also ubiquitous in mathematics, though at this level we don't bring much attention to it.

Writing in complete sentences helps you to think in complete ideas. You may not be used to writing sentences in math classes, but math is much more about being able to understand and explain ideas than just executing calculations.

2 Consider the following set of equations:

$$x + 3 = 7 \qquad x + 3 = 8 \qquad x + 3 = 9 \qquad x + 3 = 10$$

Notice that to solve all of them, you would subtract 3 from both sides of the equation. This algebraic step solves the equation regardless of what number is on the right side of the equation.

We will formally define what it means to *solve* an equation in a couple sections. See Definition 4.4.

Now consider this equation: $x + 3 = a$. We will solve this variable for x. Notice that the algebraic step remains the same. It turns out that the final result cannot be simplified any further.

$$x + 3 = a$$
$$x = a - 3 \qquad \text{Subtract 3 from both sides}$$

Try it: Solve the equation $x - 7 = c$ for the variable x using a complete presentation.

You might remember the concept of "combining like terms" for determining whether an expression can be simplified. If not, we'll get to it soon.

Your final answer will be a mathematical expression, and that's okay.

3 Consider the following set of equations:

$$x + 3 = 7 \qquad x + 4 = 7 \qquad x + 5 = 7 \qquad x + 6 = 7$$

This time, the number that is being added to x is changing. But these are still very similar equations. And the algebraic step is basically the same for each one. You need to subtract whatever number it is that's being added to the x:

$$x + 3 = 7 \longrightarrow \text{Subtract 3 from both sides}$$
$$x + 4 = 7 \longrightarrow \text{Subtract 4 from both sides}$$
$$x + 5 = 7 \longrightarrow \text{Subtract 5 from both sides}$$
$$x + 6 = 7 \longrightarrow \text{Subtract 6 from both sides}$$

Now consider this equation: $x + b = 7$. Based on the pattern above, you should have an idea of what you need to do to solve for x.

Try it: Solve the equation $x + b = 7$ for the variable x using a complete presentation.

A lot of math is seeking out and understanding patterns. This may be a simple pattern, but once you've internalized it, algebra becomes much easier.

4 The variable x is traditional, but not special. We're allowed to solve for other variables. The underlying thinking process remains the same.

Try it: Solve the equation $a + b - c = d - e$ for the variable c using a complete presentation.

You should be able to solve for any of the variables in this equation.

Understanding how to use and interpret variables is core to mathematical thinking. In contexts outside of a math class, you're often working with quantities that may be unknown to you at the moment. Being able to think mathematically will give you access to conceptualize that information in a way that gives you flexibility to respond no matter what the numbers turn out to be.

There will be situations with more complex mathematical ideas where you won't be isolating a variable by itself, but rather an entire expression. The ability to think flexibly is critical to your long-term success at mathematical reasoning. In the worksheets that follow, you will be slowly building out your capacity for working with variables to include such situations. Some of the ideas will go beyond these examples, but if you're working through the materials you should be able to see how to take each next step.

2.1 Variables in Expressions and Equations - Worksheet 1

1 Determine the value of $3x - 5$ when $x = 4$, $x = 9$, and $x = -5$. Write your results as if-then statements.

2 Solve $y + c = 7$ for y using a complete presentation.

3 Solve $x^2 + c = 7$ for x^2 using a complete presentation.

How did you solve $y + c = 7$ for the variable y? Do the same thing here.

4 Solve the equation $ax + b = c$ for the variable x using a complete presentation.

One of the biggest hurdles for students in math is a lack of self confidence to the point that they shut down without even trying. Trust yourself that you have the knowledge and experience to solve this one, and just give it a try.

2.2 Variables in Expressions and Equations - Worksheet 2

1 Determine the value of $2x - 5y$ when $x = 4$ and $y = 2$, and when $x = 2$, and $y = -3$. Write your results as if-then statements.

There is no example in the book for how to deal with two variable substitutions at the same time, but you can figure it out.

For the "then" part of your sentence, you only need to say $2x - 5y =$ (number). We'll discuss the presentation of this later.

2 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$2x + 4y = 10$$

$4y = 2x + 10$	Add $2x$ to both sides
$y = \dfrac{2x + 10}{4}$	Divide both sides by 4
$y = \dfrac{2x}{4} + \dfrac{10}{4}$	Rewrite the fraction
$y = \dfrac{x}{2} + \dfrac{5}{2}$	Reduce

Hint: In this problem, there's nothing wrong with the fraction manipulations. You can use that part of the calculation as a model for future problems.

Eventually, the dividing and rewriting should be something you can do mentally in one step. Whether you rewrite the one fraction as two separate fractions will depend on the context.

3 Solve the equation $2x + 3y = 6$ for the variable y using a complete presentation.

This is a common type of calculation when working with equations of lines. We'll see more of this later.

4 Solve the equation $2x + 3y = 6$ for the variable x using a complete presentation.

This looks a lot like the previous problem. Pay attention to the details of the wording in the instructions.

2.3 Variables in Expressions and Equations - Worksheet 3

1 Determine the value of x^2 when $x = 0$, $x = 4$, and $x = -3$. Write your results as if-then statements.

Recall that $x^2 = x \cdot x$.

What happens if you multiply a negative number by another negative number?

2 Solve $4x - 3y = 6$ for y using a complete presentation.

Be careful with your negative signs. Remember that $\frac{a}{-b} = -\frac{a}{b}$.

3 Solve $ax + by = c$ for y using a complete presentation.

4 Work backwards from the given information to derive the original presentation.

Try not to feel overwhelmed. If you've done the worksheets up to this point, you can do this one.

	Add c to both sides
$a = \dfrac{b+c}{d}$	Divide both sides by d

2.4 Variables in Expressions and Equations - Worksheet 4

1 Determine the value of $x^2 + 2y^2$ when $x = -1$ and $y = 2$, and when $x = -2$ and $y = -1$. Write your results as if-then statements.

2 Solve $ax + by = c$ for x using a complete presentation.

3 Solve the equation $3x - 2y - 4z = 24$ for the variable z using a complete presentation.

This is not dramatically different the two variable problems. Trust yourself and give it a try.

2.5 Variables in Expressions and Equations - Worksheet 5

1 Solve the equation $(x - 5) + a = b$ for the expression $(x - 5)$.

This may seem odd right now, but being able to solve for a complicated variable expression is an important skill to develop.

2 Solve the equation $3(y + 3) + a = b$ for the expression $(y + 3)$.

Think of the entire set of parentheses as a single object. Do not distribute the 3.

3 Work backwards from the given information to derive the original presentation.

This is a challenging problem! You may want to look back at some of the problems on previous worksheets in this section.

When you're done, you should check your work by reading it from top to bottom.

	Add 4 to both sides
	Divide both sides by 6
	Rewrite the fraction
$a = \dfrac{b}{6} + \dfrac{2}{3}$	Reduce

Hint: The $\frac{2}{3}$ was the part that was reduced.

2.6 Deliberate Practice: Solving Equations for a Variable Expression

Focus on these skills:

- Write the original equation.
- Line up your equal signs.
- Correctly state the algebraic steps using the correct phrasing:
  - Add (expression) to both sides.
  - Subtract (expression) from both sides.
  - Multiply both sides by (expression).
  - Divide both sides by (expression).
- Execute the algebra and arithmetic correctly.
- Present your work legibly.

Instructions: Solve the equations for the variable expression using a complete presentation.

1 Solve $6x^2 + 23 = 41$ for the expression x^2.

2 Solve $2y^3 + 14 = -20$ for the expression y^3.

3 Solve $5(t - 5) - 7 = 7$ for the expression $(t - 5)$.

4 Solve $6(r + 8) - 18 = 53$ for the expression $(r + 8)$.

5 Solve $3(b^2 + 4) - 25 = -38$ for the expression $(b^2 + 4)$.

6 Solve $2x^2 + 4y^2 + 3z^2 + 5 = 11$ for the expression x^2.

7 Solve $2x^2 + 4y^2 + 3z^2 + 5 = 11$ for the expression y^2.

8 Solve $2x^2 + 4y^2 + 3z^2 + 5 = 11$ for the expression z^2.

9 Solve $4\ln(x) + 7 = 22$ for the expression $\ln(x)$.

You don't need to know what $\ln(x)$ or $\sin(2x)$ mean to do these last two problems.

10 Solve $3\sin(2x) + 8 = 14$ for the expression $\sin(2x)$.

2.7 Closing Ideas

Most textbooks will have hundreds of problems for you to work on, especially for a section like this one. And there's certainly value to practicing those manipulations over and over again so that you can get a lot of experience. However, the approach of this textbook is much more about slowing you down to help you create a solid mental framework for understanding mathematical ideas.

One piece of that framework of mathematical thinking is the ability to take something complicated and break it down into simpler pieces. One of the ways that manifests is the transition from working with numbers to working with variables. We tried to set you up with this idea in the forward direction by giving you a series of equations with numbers before asking you to work with variables:

$$\left.\begin{array}{l} x+3=7 \longrightarrow \text{Subtract } 3 \text{ from both sides} \\ x+4=7 \longrightarrow \text{Subtract } 4 \text{ from both sides} \\ x+5=7 \longrightarrow \text{Subtract } 5 \text{ from both sides} \\ x+6=7 \longrightarrow \text{Subtract } 6 \text{ from both sides} \end{array}\right\} \implies x+b=7 \longrightarrow \text{Subtract } b \text{ from both sides}$$

If you run into a situation where you're not entirely sure what to do, you can always replace the variable with numbers to help you think about it.

$$x+b=7 \implies \begin{cases} x+3=7 \\ x+4=7 \\ x+5=7 \\ x+6=7 \end{cases}$$

Underneath this idea is the fact that variables represent numbers. And once you learn to think about variables as if they are numbers, the algebra becomes less confusing and less intimidating.

2.8 Going Deeper: Sets and Solution Sets

In mathematics, a set is just a collection of objects. In a very real way, sets are among the most fundamental objects in mathematics. There is a particular type of set that is associated with equations involving variables, which are known as solution sets. Basically, a solution set is the set of all the numbers that you can plug into the equation to get a valid if-then statement regarding the values.

For example, the solution set of the equation $x^2 = 4$ is $\{-2, 2\}$. We can check this by thinking through the appropriate if-then statements:

- If $x = -2$, then $x^2 = 4$.
- If $x = 2$, then $x^2 = 4$.

Notice that both of these statements are true. But we need to go further to justify that all other numbers fail to make a true statement. For example, "If $x = 5$, then $x^2 = 4$" is false since $5^2 = 25$ and 25 is not the same as 4. But checking that one number isn't good enough because there are infinitely other numbers to check (including all decimals and fractions). We clearly can't check an infinite number of values, and so we need some sort of logical structure to explain how we can know that we've got all the solutions.

What is that logical structure? Algebra!

Notice in the naming of the set, we used the symbols { and }. These are known as set brackets. You can think of the set brackets as a bag in which you're going to put various objects. The contents in between the set brackets describe the objects in the set in some form. In our example, we presented that as a list. We would read $\{-2, 2\}$ as "the set containing the numbers -2 and 2." We don't really care about the order that we list the values, so that this set is the same as $\{2, -2\}$.

Some books go into some detail about *set-builder notation*, which is another way of describing sets. Rather than making an explicit list of elements, the set is defined by properties. Here is an example:

$$\{x \in \mathbb{R} : x^2 = 4\}$$

Some people use | instead of : for set-builder notation.

You can immediately see that set-builder notation has a lot of symbols. We'll explain how to read this and then break it down into components.

$$\begin{array}{ccccc} \{ & x \in \mathbb{R} & : & x^2 = 4 & \} \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{The set of} & \text{real numbers } x & \text{such that} & x^2 = 4 & \end{array}$$

What are the real numbers? That question is much more complicated than you think. For now, just think of them as all possible numbers.

We've already talked about the set brackets, so let's look at the other pieces.

- $x \in \mathbb{R}$: This defines two objects. The first object it defines is a variable. This is a symbol that represents the quantity with the desired property. In this case it's the symbol x, but it can be basically any symbol you want. The second object that this defines is the *universal set*. This is important because it defines the collection of objects that we're going to consider. In this case, we're thinking about real numbers. If you happen to be familiar with complex numbers, you will probably remember that there is a special number called i that exist in the complex numbers that isn't part of the real numbers. For this situation, we're saying that we're not considering complex numbers to be options for our variable.

The symbol i is called the *imaginary unit* and represents the quantity $\sqrt{-1}$.

- $x^2 = 4$: This is the property that the variable needs to satisfy. In other words, in order to be in the set the variable must satisfy this equation. This part of the notation can be a single equation, multiple equations, or even a collection of words that describe the property.

This set is the solution set of the equation $x^2 = 4$ because it says that it is the set of all real numbers that make $x^2 = 4$ a true equation. And this shows one of the downsides of this notation. When we define sets in terms of properties, it may still leave us with work to do to figure out what values actually have the desired property. In order for us to know that this set contains the numbers -2 and 2, we would still have to do some work. And that's both the strength and weakness of set-builder notation.

On the one hand, the notation is flexible enough that we can defined objects by a list of properties. On the other hand, that flexibility sometimes means that even though we know what properties the objects should have, we often don't actually know what the exact objects are that are in the set.

Some books try to teach students set-builder notation and ask them to present their solutions to equations as sets. Unfortunately, this is makes things even more confusing and pushes students deeper into the rule of rule following rather than understanding. This puts students either in the position where their notation is wrong, or their notation is correct but they don't really understand what they're writing. Neither is a good thing.

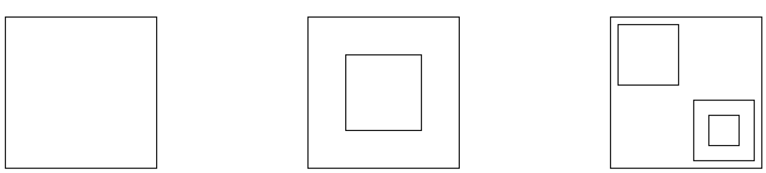
Set theory is the mathematical field that studies sets very carefully. It needs to be done carefully because as simple as it may appear here, it turns out to be an incredibly difficult subject. We won't be able to explore the details of that, but we can get a brief glimpse of some of the complexity that arises from that area.

Let's start with a set that contains nothing. There are two notations for this: $\{\}$ (literally, a set with nothing in it) and $\emptyset$. The latter notation is far more common, so that's what we will use. We can think of this as being like an empty box. It's a collection of objects where there is nothing inside.

But now let's consider this object: $\{\emptyset\}$. This is the set that contains the empty set. This is mathematically different from the empty set in the same way that a box containing an empty box is not the same thing as an empty box. And you can already start to see some of the difficulty of this area of mathematics. You have to parse the symbols very carefully to understand what's happening, because even though the object inside the box is an empty box, we have to remember that the outside box is not actually empty.

Or consider this object: $\{\emptyset, \{\emptyset\}\}$. This is the set that contains the empty set and the set that contains the empty set. In other words, it's a box with an empty box and a box containing and empty box. We might want to ask a basic question: How many objects are in this set? And this turns out to be somewhat challenging because there are actually four boxes in the picture now. What is the "correct" way of counting them?

It turns out that mathematicians have decided that the answer should be two. When you have containers inside of containers, it's better to not have to worry about what's inside the inner containers.



$\emptyset$ $\quad\quad$ $\{\emptyset\}$ $\quad\quad$ $\{\emptyset, \{\emptyset\}\}$

While this may appear to be extremely abstract, this way of thinking turns out to be very closely related to how huge amounts of data is organized. For example, phones and computers use a file folder structure that looks like boxes inside of boxes, and the internet uses a lot of JSON data which can also be thought of as boxes inside of boxes. And so the abstract ideas of set theory end up having practical ways of being interpreted and applied.

2.9 Solutions to the "Try It" Examples

1

If $x = 5$, then $2x = 10$.
If $x = 12$, then $2x = 24$.
If $x = -3$, then $2x = -6$.

2

$x - 7 = c$	
$x = c + 7$	Add 7 to both sides

3

$x + b = 7$	
$x = -b + 7$	Subtract b from both sides

$x = 7 - b$ is also acceptable.

4

$a + b - c = d - e$	
$b - c = -a + d - e$	Subtract a from both sides
$-c = -a - b + d - e$	Subtract b from both sides
$c = a + b - d + e$	Multiply both sides by -1

It is often helpful to keep your variables in alphabetical order.

One of These Things is Not Like the Other: Like and Unlike Terms

Learning Objectives:

- ☐ Identify the coefficient and variable part of a monomial.
- ☐ Identify the monomials of a polynomial.
- ☐ Determine whether terms in a mathematical expression are like terms or unlike terms.
- ☐ Simplify expressions by combining like terms.

The concept of combining like terms is not that different from the way we naturally organize information in our heads. Consider the following question: If you have one bag that contains two apples and three oranges and you had a second bag that has four apples and one orange, how many of each fruit do you have?

It should only take a moment's thought to conclude that you have six apples and four oranges. We can write this out in a mathematical notation as follows:

Did you think about the question before moving on? Or did you just keep on reading without stopping? When books ask you a question like this, the hope is that you would think about it for yourself because that helps you to understand the ideas better.

$$(2 \text{ apples} + 3 \text{ oranges}) + (4 \text{ apples} + 1 \text{ orange}) = 6 \text{ apples} + 4 \text{ oranges}$$

There's a lot to say here about the notation:

- Notice how the parentheses serve as natural boundary markers for each bag. By looking at the mathematical notation, you know exactly what's contained in the first bag and what's contained in the second bag.

Parentheses are called "grouping symbols" because they group different types of objects together.

- Notice how the addition sign corresponds to your intuition about addition. You can rearrange things so that the first set of apples are with the other set of apples, and the same thing with the oranges.
- Notice how you have no impulse to try to combine the apples and oranges together to get "10 apple-oranges" as your final answer.

You can conclude that there are 10 pieces of fruit, but that doesn't answer the question.

This simple word problem serves as an illustration for how we think about combining like terms. It's easier for us to see what's going on here because we have some experience handling pieces of fruit and we can easily recognize that one type of fruit is different from the other. The same concept applies to algebraic expressions, but we first need to make sure we know what we're looking at.

A shorter example of this is that one apple and one tree does not give you two apple trees. But that doesn't let us explore the notation as much.

Definition 3.1. A *monomial* is any product of numbers and variables. The *coefficient* is the product of all of the numbers and the *variable part* is the product of all the variables.

Although it's possible for us to write a monomial as $3 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y$, we usually take advantage of exponent notation to write it as $3x^4y^2$ to keep things compact. This also makes it much easier to read. In this example, the coefficient is 3 and the variable part is x^4y^2.

The degree of a monomial is the number of variables in the product when we write it out in the expanded form. So $3x^4y^2$ has degree 6.

There are a couple special situations to remember:

- If you have a number by itself (such as the number 12), the term is called a constant and it is considered to have no variable part.
- If you have a variable by itself (such as with x or y^2), the coefficient is implicitly a 1. So it would be correct to think of x as $1x$ and y^2 as $1y^2$.

Mathematicians prefer an economy of notation, so symbols like that are dropped when they're easily understood.

Definition 3.2. A *polynomial* is the sum of a number of monomials.

We allow for monomial by itself to be a polynomial because it would be annoying if we didn't. The use of an inclusive terminology greatly simplifies things for us.

The definition says that a polynomial is a sum of monomials, not a difference of them. But we will often write polynomials with subtraction, such as with $x^2 - 6x + 8$. The reason for this is that we can use the idea that subtraction is addition of the opposite to rewrite it:

$$x^2 - 6x + 8 = x^2 + (-6x) + 8$$

One of the features of writing subtraction as addition is that it visually groups the negative sign with the monomial. This means that when we identify the coefficient of a monomial term, we need to account for the negative sign. Therefore, the coefficient of the x term is -6.

1 Understanding vocabulary is a multi-step process. It's not enough to just read the definitions and see a couple examples. In order for the definition to sink in effectively, you must take the time to work on problems yourself.

The entire book is structured with this perspective in mind.

Try it: Using the grid below, identify all of the monomials of $x^2 + 7x - y - 8$. Then determine the coefficient and variable part of each term.

For terms with no variable part, writing NA in the box is good enough. Make sure you pay attention to the subtraction symbols and negative signs.

Monomial				
Coefficient				
Variable Part				

Definition 3.3. Two monomials are *like terms* if they have the same variable part. If their variable parts are different, then they are *unlike terms.*

You probably could have guessed the definition of unlike terms by yourself. But mathematicians like to be thorough.

2 Combining like terms is an algebraic process that uses the distributive property, but it's backwards from how it's most commonly seen. This reverse distributive property is often called "factoring out" terms. When we have like terms, we can perform the arithmetic on the coefficients to simplify the expression.

$$\begin{aligned} 5x + 8x &= (5+8)x && \text{Factor out the } x \\ &= 13x && \text{Arithmetic} \end{aligned}$$

Notice that the presentation follows the same general scheme as before. The equal signs are lined up and the explanation is on the right

Notice how natural this is based on the logic of combining either the apples or the oranges.

If you have 5 x-es and you add 8 more x-es, how many x-es do you have? 13 x-es.

Try it: Simplify the expression $7y^2 + 6y^2 - 5y^2$ using a complete presentation. Show the step where you factor out the common factor.

There's no example with three terms, but you can figure it out. Trust yourself!

3 What happens if there are unlike terms? Once again, we can think about the apples and oranges. Basically, we're just stuck with what we have. We've actually been using this already. In previous sections, if we had an expression like $4x + 3$, we just left it like that. And this was also true in the previous section where the different variables were all left separate.

But what if there are some terms are like terms and others are unlike terms? Think back to the apples and oranges. What happened there? In that case, the apples were combined and the oranges were combined, but the fruits remained separate. We do the same thing in algebra.

$$\begin{aligned}
(5x - 3y) + (4x + y) &= 5x + 4x - 3y + y && \text{Rearrange the terms} \\
&= (5x + 4x) + (-3y + y) && \text{Group the terms} \\
&= (5 + 4)x + (-3 + 1)y && \text{Factor out the } x \text{ and } y \\
&= 9x - 2y && \text{Arithmetic}
\end{aligned}$$

This is the dangerous step. Be careful!

We normally don't show this step. We're just doing it here for emphasis.

Students are sometimes not very careful with parentheses. They often write the wrong thing even if what's in their head is correct. Remember that the correct communication of ideas is important in math, so it's not good enough to write the wrong thing and pretend that it's correct. Here are two very common errors that students make:

$$5x + 4x - 3y + y \stackrel{\times}{=} (5x + 4x)(-3y + y)$$
$$5x + 4x - 3y + y \stackrel{\times}{=} (5x + 4x) - (3y + y)$$

The first one happens when students simply draw parentheses around symbols without thinking about how it affects the meaning. Notice that the right side is a product of terms because the parentheses are right next to each other. The second one happens when students don't pay attention to the fact that the negative sign is part of the coefficient of the $-3y$ term. These errors are avoidable with practice. But it takes practice because you need to train you brain to see the expressions differently.

When parentheses are next to each other like that, we call it "implied multiplication."

Try it: Simplify the expression $(-4a - 3b) + (-5a + 8b)$ using a complete presentation. Show all the steps that are in the example above.

Remember that the negative sign stays with the term!

So far, when we have combined like terms, we've been focusing on addition. But we also need to be able to do this with mathematical expressions that have subtraction and multiplication. For each of these, we need to remember to use the distributive property.

The idea behind this one comes from the concept of multiplication representing groups of things. If you have four bags that each contain two apples and three oranges, how much fruit do you have in total?

$$4 \cdot (2 \text{ apples} + 3 \text{ oranges}) = 8 \text{ apples} + 12 \text{ oranges}$$

We have used the multiplication dot to emphasize the multiplication.

This is a demonstration of an important property of algebra.

Definition 3.4. The *distributive property* states that $a(b+c) = ab + ac$. The full name of this the *distributive property of multiplication over addition.*

4 You might recall that in the order of operations, multiplication comes before addition and subtraction. This means that you should use the distributive property before you combine like terms.

The distributive property is a multiplication operation and combining like terms is an addition and subtraction operation.

$$\begin{aligned}
(2x-5y)+3(x-2y) &= (2x-5y)+(3x-6y) && \text{Distributive property}\\
&= 2x+3x-5y-6y && \text{Rearrange the terms}\\
&= (2x+3x)+(-5y-6y) && \text{Group the terms}\\
&= (2+3)x+(-5-6)y && \text{Factor out the } x \text{ and the } y\\
&= 5x-11y && \text{Arithmetic}
\end{aligned}$$

Try it: Simplify the expression $(3a+2b)+2(5a-3b)$ using a complete presentation. Show the distributive step, the rearrangement step, the grouping step, and the arithmetic step.

5 Whenever we have an algebraic symbol next to a set of parentheses with multiple terms on the inside, we need to use the distributive property. When we were doing it earlier with addition, we basically just multiplied by 1, which is why there was no change. If the operation outside of the parentheses is subtraction, then we also need to distribute that negative sign.

Remember that multiplying by 1 does not change the value.

To understand why the negative sign should be distributed, we will think about another example using apples and oranges. Suppose you have eight apples and six oranges, and someone takes four apples and five oranges away from you. How much fruit do you have left?

$$(8 \text{ apples} + 6 \text{ oranges}) - (4 \text{ apples} + 5 \text{ oranges}) = 4 \text{ apples} + 1 \text{ orange}$$

Notice that in context, it is completely natural to remember to subtract off the 5 oranges from the 6 oranges. However, when manipulating symbols that context is easily lost, which leads to errors like this one:

$$(8x+6y)-(4x+5y) \overset{\times}{=} 8x+6y-4x+5y$$

This is an *extremely* common error!

Try it: Simplify the expression $(8x+6y)-(4x+5y)$ using a complete presentation. Show the distributive step, the rearrangement step, the grouping step, and the arithmetic step.

Notice how these words match up with the presentation in the previous example. Language is important in mathematics because it helps us to communicate our ideas precisely.

3.1 Like and Unlike Terms - Worksheet 1

1 Using the grid below, identify all of the monomials of $4x^3 - 3x^2 + 6x - 5$. Then determine the coefficient and variable part of each term.

Monomial				
Coefficient				
Variable Part				

Mistakes involving negative signs and subtraction symbols are among the most common mistakes for students to make. Part of the value of this exercise is to train your mind to pay close attention to those symbols.

2 Simplify the expression $(5a + 3b) + (8a - 5b)$ using a complete presentation. Show the rearrangement step, the grouping step, and the arithmetic step.

We are asking you to show all of the steps because it's important to be able to communicate these manipulations clearly. Use these instructions like a checklist of algebraic manipulations that you should know how to do.

3 Two students perform the calculation $6x - 6x$. One student claims that the answer is x while the other claims the answer is 0. Determine which student did the calculation correctly and explain your reasoning using complete sentences. Then explain the error that the other student made.

Knowing why certain manipulations are wrong can be as helpful as knowing why certain manipulations are right.

3.2 Like and Unlike Terms - Worksheet 2

1 Using the grid below, identify all of the monomials of $3x^3 - 5xyz + xz - 3$. Then determine the coefficient and variable part of each term.

Monomial				
Coefficient				
Variable Part				

2 Simplify the expression $(7m + 3n) + (5m - 3n)$ using a complete presentation. Show the rearrangement step, the grouping step, and the arithmetic step.

3 Simplify the expression $(5x + 3) - (2x - 5)$ using a complete presentation. Show the distribution step, the rearrangement step, the grouping step, and the arithmetic step.

Remember that a negative multiplied by a negative is a positive.

3.3 Like and Unlike Terms - Worksheet 3

1 Simplify the expression $(3a + 2b) + 2(-a + 3b)$ using a complete presentation. Show the distribution step, rearrangement step, the grouping step, and the arithmetic step.

2 Simplify the expression $(-6x + 4y) - (-6x - 4y)$ using a complete presentation. Show the rearrangement step, the grouping step, and the arithmetic step.

Be careful!

3 Simplify the expression $(7x - 5) - (3x - 4)$.

Although it's important to be able to explain every step carefully, in practice we don't always do that. Your instructor may or may not have specific instructions for what they are looking for. Just remember that the main thing is that your work must be legible and understandable to someone else.

3.4 Like and Unlike Terms - Worksheet 4

1 Simplify the expression $2(4a^2 + 5ab - 3b^2) - (3a^2 - ab + 7b^2)$.

You were never told how to handle three different types of terms. But if you understand what you were doing before, you should be able to do this without extra guidance.

2 Simplify the expression $3(x^2 - 4x + 2) - 2(x^2 - 5)$.

3 Two students perform the calculation $5x^2 - x^2$. One student claims that the answer is $4x^2$ while the other claims the answer is 5. Determine which student did the calculation correctly and explain your reasoning using complete sentences. Then explain the error that the other student made.

3.5 Like and Unlike Terms - Worksheet 5

1 Simplify the expression $-2(3m^2 - 6mn + 4n) + 3(-m^2 + 4mn + 3n)$.

2 Simplify the expression $3(x^2 - 5x + 4) - 2(3x^2 - 5x + 6)$.

3 Simplify the expression $3(t^2 - 6t + 2) - 2(t^2 + 4t) + 3(t - 5)$.

3.6 Deliberate Practice: Combining Like Terms

Focus on these skills:

- Write the original expression.
- Line up your equal signs.
- Show the distribution step, the rearrangement step, the grouping step, and the arithmetic step.
- Be careful with negative signs and the distributive property.
- Present your work legibly.

Instructions: Simplify the expression.

1. $(3x + 5y) + 2(6x - 4y)$

2. $2(4a - 3b) - 3(-3a - 2b)$

3. $-3(-4m + 2n + 4) + 2(3m - 3n + 7)$

4. $4(-3r + 7s - 3) - 3(2r - 3s - 4)$

5. $-4(5x + 3y - 2z) - (-3x + 4y + 3z)$

6. $3(r^2 + 2s^2 - 4) + 5(-2r^2 + s^2 - 3)$

7. $-2(3x^2 + 7xy - 4y^2) + 3(2x^2 - 3xy + 5y^2)$

8. $4(2m - 5n + 1) - 2(-3m + 2n - 3) + 5(m + 4n - 2)$

9. $4(x^2 - 4x + 1) + 3(-2x^2 + 3x - 4) - (-2x^2 - 5)$

10. $2(r^3 - 3r^2s + s^3) - 3(2r^3 + 3rs^2 - 5s^3) + 2(r^2s - rs^2)$

Pay attention to the details!

3.7 Closing Ideas

In this section, we rearranged terms many different times and in many different ways. As long as we made sure that the negative signs moved with the appropriate terms, everything worked out just fine. But we didn't really discuss why things worked out fine. We just showed you how to do it, and then let you mimic that.

There are some fundamental properties of addition that you've probably seen at some point before.

Definition 3.5. Let a, b, and c be real numbers. Then the following properties hold:

- *The Commutative Property of Addition*: $a + b = b + a$
- *The Associative Property of Addition*: $(a + b) + c = a + (b + c)$

Notice that this property does not apply to subtraction. You should be able to see that $a - b$ and $b - a$ are not the same value. If we were to switch their positions without changing the value, we would first have to rewrite it as addition, and then apply the commutative property of addition to swap their positions. If you've worked through the exercises, then this should feel very familiar.

Use the ideas from the previous section and replace the variables with numbers to check this!

$$\begin{aligned} a - b &= a + (-b) && \text{Subtraction is addition of the opposite} \\ &= (-b) + a && \text{Commutative property of addition} \end{aligned}$$

As it turns out, these two addition properties are what give us the ability to move things around the way we have this section. The reason that we need these properties is extremely subtle. Have you ever noticed that you can only add two numbers at a time? This doesn't mean that you haven't seen sums like $3 + 5 + 5$ before, but when you actually go to calculate this, you don't actually work with all of the numbers all at once.

We call addition a binary operation because it only deals with two numbers at a time.

This means that when we do arithmetic, we are always implicitly putting parentheses all over the place. If you're just working from left to right, then you would see the calculation as $(3 + 5) + 5$. That is, you add the 3 and the 5 first, and then you add 5 to that result.

But maybe you've got a bit of intuition and recognize that $5 + 5 = 10$, which is potentially an easier or faster approach. In that case, you would see the calculation as $3 + (5 + 5)$. This is not a problem because the result is the same both ways!

The commutative property is also used when doing calculations. For most people, the calculation $27 + 2$ is much easier to think about than $2 + 27$. And so when we see that calculation, we instinctively switch it around to the way that's easier for our brains to think about. And because of the commutative property, it doesn't change the result.

The reason $27 + 2$ is easier to think about than $2 + 27$ is because we often think of addition by starting from the first number and counting up by the second one. Count up 2 is much easier than counting up 27.

3.8 Solutions to the "Try It" Examples

1

Monomial	x^2	$7x$	$-y$	-8
Coefficient	1	7	-1	-8
Variable Part	x^2	x	y	NA

2

$$\begin{aligned} 7y^2 + 6y^2 - 5y^2 &= (7+6-5)y^2 && \text{Factor out the } y^2 \\ &= 8y^2 && \text{Arithmetic} \end{aligned}$$

3

$$\begin{aligned} (-4a-3b)+(-5a+8b) &= -4a-5a-3b+8b && \text{Rearrange the terms} \\ &= (-4a-5a)+(-3b+8b) && \text{Group the terms} \\ &= (-4-5)a+(-3+8)b && \text{Factor out the } a \text{ and the } b \\ &= -9a+5b && \text{Arithmetic} \end{aligned}$$

4

$$\begin{aligned} (3a+2b)+2(5a-3b) &= (3a+2b)+(10a-6b) && \text{Distributive property} \\ &= 3a+10a+2b-6b && \text{Rearrange the terms} \\ &= (3a+10a)+(2b-6b) && \text{Group the terms} \\ &= (3+10)a+(2-6)b && \text{Factor out the } a \text{ and } b \\ &= 13a+4b && \text{Arithmetic} \end{aligned}$$

5

$$\begin{aligned} (8x+6y)-(4x+5y) &= 8x+6y-4x-5y && \text{Distributive property} \\ &= 8x-4x+6y-5y && \text{Rearrange the terms} \\ &= (8x-4x)+(6y-5y) && \text{Group the terms} \\ &= (8-4)x+(6-5)y && \text{Factor out the } x \text{ and the } y \\ &= 4x+y && \text{Arithmetic} \end{aligned}$$

You can write your final result as $4x + 1y$ if you want, but you will almost always see that written with the implicit coefficient.

Read the Instructions: Simplifying Expressions and Solving Equations

Learning Objectives:

- ☐ Distinguish between mathematical equations and expressions.
- ☐ Recognize the differences between different sets of instructions.
- ☐ Correctly execute the instructions that are given.

An important aspect of mathematics is the attention to detail and precision required to do it well. However, for many students (and even many teachers) the importance of that precision is often overlooked in favor of repetitive execution. The most prevalent example of imprecision in language for mathematics at this level is the word "solve." Unfortunately, this word is used as a placeholder for a wide range of mathematical procedures. Here are some examples:

Words are extremely important to mathematics.

- Solve $5 + 8$.
- Solve $5x + 8x$.
- Solve $5x + 8 = 23$.
- Solve $x^2 + 6x + 9$.

Only one of these reflects the proper usage of the word "solve." Do you know which one it is? Can you come up with a description of what that word means?

Are you going to stop to think about that question, or just keep reading?

When one word comes to represent so many different activities, it becomes extremely confusing to know what's being asked of you. But by taking the time to be precise with our language, we can create mental categories to help us keep information organized. Here are the same calculations, but given with proper instructions.

- Calculate $5 + 8$.
- Simplify $5x + 8x$.
- Solve $5x + 8 = 23$.
- Factor $x^2 + 6x + 9$.

There are even more words that can be used as instructions. Evaluate, compute, estimate, round, rewrite, verify, prove, plot, sketch, and graph are just some of them. Some of these are similar to each other, and some are very different.

Do any of these words make you think of specific procedures?

The purpose of this section is to help you to make sense of some of these words to create some of the mental structures that will help you think through the ideas more effectively. As usual, we will start with a couple definitions.

Definition 4.1. A mathematical *expression* is a meaningful collection of mathematical symbols that represents a value.

Whether or not something is meaningful depends on your level of knowledge. And the number of symbols that are meaningful should grow over time in the same way that your vocabulary grows over time. At this point, you should recognize that "$5+$" is not meaningful, but you might not be able to determine whether "$2\sin\pi - 1$" is if you've never seen trigonometry (and maybe even if you have).

"$5+$" doesn't say what number to add to 5. "$2\sin\pi - 1$" is a valid expression. It's okay if you don't know what that means right now.

Numbers (such as 5 and $\sqrt{2}$) and monomials (such as x and $-6y^2$) are expressions. Expressions may also consist of calculations (such as $5+8$) or polynomials (such as $x^2 - 8x + 12$). Between these last two examples, there is an important distinction. There is a shorter way to write $5+8$ with fewer symbols (namely, as the number 13), but we cannot do the same with $x^2 - 8x + 12$.

> **Definition 4.2.** To *simplify* an expression means to find the simplest mathematical expression that represents the same quantity.

This is a broad category for a lot of mathematical concepts. For example, the process of combining like terms is one way to simplify an expression, since we can reasonably see that $5x + 8x$ is not as simple as $13x$. We can also use this word for arithmetic calculations such as $8+6\cdot 7-12$. You will sometimes see words like calculate or compute for these types of problems.

We will take a deeper look at the order of operations in a later section.

We didn't bring any attention to it, but in the last section we demonstrated the presentation format for problems where the goal is simplification.

$$\begin{aligned}
\text{(Original Expression)} &= \text{(Manipulated Expression 1)} && \text{Explanation 1} \\
&= \text{(Manipulated Expression 2)} && \text{Explanation 2} \\
&= \cdots \\
&= \text{(Final Expression)} && \text{Final explanation}
\end{aligned}$$

You can think of this as a long line of small manipulations. It's helpful to go over to the right before going down so that you can more easily visualize the idea that the top expression on the left side is the thing that is equal to all of the expressions on the right. But it's not strictly necessary to do that. The more important thing is to recognize that each new line is a continuation from the previous one, and not an entirely new equation.

> 1
>
> If you look back at the section on combining like terms, you will see the above presentation in all of the presented work throughout the section. The examples exist not only to show you the ideas, but also to model how you write things.
>
> **Try it: Simplify the expression $(3x+7y)-(4x+5y)$ using a complete presentation.**

The reason the distinction in presentation is important is because the types of things you're allowed to do when simplifying an expression are different from the things you can do when solving an equation.

> **Definition 4.3.** A mathematical *equation* is a statement of the form $A = B$, where A and B are mathematical expressions. The statement $A = B$ means that both A and B represent

the same mathematical quantity.

The equal sign can be read as "represents the same quantity as," so that $A = B$ can be read as "A represents the same quantity as B."

Some people use "is" because it's a lot shorter. However, it's a little less precise to say it that way.

If we think of expressions as phrases, then equations are sentences. The important distinction here is that equations make a declaration that is either true or false, where as expressions are just ideas. For example, "the dog" is just an expression. If we say "the dog" without any context, the response is, "What about it?" But we can turn it into a complete thought, such as "the dog is brown," which is going to be either true or false. We won't know whether it's true or false until we pick a particular dog to look at.

In the same way, x is a mathematical expression. It just represents a number, like 5. It's not until we put it into a mathematical equation (such as $2x = 10$) that we can start to analyze the truth value. Depending on the specific value of x, the equation might be true or it might not be true. If $x = 4$ then the equation is not true, but if $x = 5$ then the equation is true.

Definition 4.4. To *solve* an equation means to find the value (or values) of the variable (or variables) that make the equation true. A *solution* to an equation is a specific value of the variable (or specific values of the variables) that make the equation true.

We're not going to spend a lot of time discussing *truth values* and just rely on your intuition. The equation $2 + 3 = 5$ is true and the equation $7 - 4 = 2$ is false. We can apply the same ideas to equations with variables. It's true that if $x = 2$ then $3x - 1 = 5$, and it's false that when $x = 2$, $3x - 1 = 7$. This means that $x = 2$ is a solution of the equation $3x - 1 = 5$, but is not a solution of the equation $3x - 1 = 7$.

When working with equations, the key idea is that we have to maintain the equality. If we perform an arithmetic operation on one side of the equation, then we must do it to the other side. As you get further in mathematics, the types of manipulations become more complex, and there are other types of properties that you have to keep in mind when working with equations.

For example, if you take the square root of both sides of an equation, you need to use a $\pm$ symbol since that operation creates two possibilities.

The presentation format when working with equations looks like the following:

(Original Equation)	
(Manipulated Equation 1)	Explanation 1
(Manipulated Equation 2)	Explanation 2
...	
(Final Equation)	Final explanation

When reading the meaning of this, what we're saying is that if the original equation is true, then the first manipulated equation is true because of the reason in explanation 1. Then the next equation is true because of the next manipulation, and so forth.

2

If you look back at the section about algebraic presentation, we used the same structure there that we're advocating here.

Try it: Solve the equation $5x - 7 = 7x + 11$ using a complete presentation.

4.1 Simplifying Expressions and Solving Equations - Worksheet 1

1 Simplify the expression $2(7a - 4b) - 4(2a - b)$ using a complete presentation. Show the distribution step, the rearrangement step, the grouping step, and the arithmetic step.

2 Look through your presentation on the previous problem. Which steps do you think can most safely be skipped in terms of demonstrating your understanding? In terms of avoiding errors? Explain your reasoning.

3 Simplify the expression $3(x^2 - 2x + 4) - 2(2x^2 + 5)$ using a complete presentation. Show only the steps that you think are the important.

Be sure to pick an appropriate level of explanation for the presentation you're providing.

Note that your instructor may set specific expectations for your presentation. If they do that, then do what they tell you to do. Just remember that the real goal is to understand and be able to communicate that understanding effectively.

4.2 Simplifying Expressions and Solving Equations - Worksheet 2

1 Determine whether $p = -3$ is a solution of the equation $-2p - 4 = -2$.

In other words, determine if the two sides of the equation give the same value when $p = -3$.

2 Solve the equation $7x = x + 24$ using a complete presentation.

3 Consider the following presentation for solving the equation from the previous problem.

$7x = x + 24$	
$7x - x = x + 24 - x$	Subtract x from both sides
$7x - x = x - x + 24$	Rearrange the terms
$7x - x = (x - x) + 24$	Group the terms
$(7 - 1)x = (1 - 1)x + 24$	Factor out the x
$6x = 0x + 24$	Arithmetic
$6x = 24$	Simplify
$\frac{6x}{6} = \frac{24}{6}$	Divide both sides by 6
$x = 4$	Arithmetic

The work above is all technically correct. Why do you think this would be considered a problematic presentation?

Go beyond the idea that this it is too long. Think about the goals of the problem and how the work relates to that goal.

4.3 Simplifying Expressions and Solving Equations - Worksheet 3

1 Determine whether $q = -2$ is a solution of the equation $3q + 1 = -q - 7$.

2 Solve the equation $3(y + 4) - 2 = 2y - 7$ using a complete presentation.

Hint: Find a way to simplify to left side of the equation by writing it without parentheses.

3 Solve the equation $3(2n - 1) + 5 = 4n - 8$ using a complete presentation.

4 Simplify the expression $4(a^2 - 2ab + 3b^2) - 3(a^2 + 4ab) - 3(ab + 2b^2)$.

4.4 Simplifying Expressions and Solving Equations - Worksheet 4

1 Verify that $y = 4$ and $y = -4$ are both solutions of the equation $-y^2 + 6 = -10$.

To verify the solution means to show by direct calculation that the equation is true. It's similar to asking you to determine whether a given value is a solution, except that you already know that the calculation is supposed to show that it is a true equation.

2 For the previous problem, what mistake do you think would be common for students to make?

3 Solve the equation $5t + 8 = -3t + 15$ using a complete presentation.

4 Solve the equation $-2(s - 3) + 2 = 2s + 8$ using a complete presentation.

4.5 Simplifying Expressions and Solving Equations - Worksheet 5

1 Solve the equation $5(2a+7)-3a+4=3(a-3)-(2a+1)$ using a complete presentation.

2 There is no value of x that makes the equation $3x+5=5(x+1)-2x$ false. This means that every choice of the variable x will result in a true equation. Attempt to solve the equation using the normal method. Describe what your equation looks like and why it makes sense to conclude that all values of x make the equation true.

The situation described in this problem and the next often gets students confused. The point of this is to remind you what the meaning of these equations are so that you aren't just blindly memorizing more rules. It's important to understand the underlying logic of the problem.

3 There is no value of x that makes the equation $-2x+3=-2(2x+1)+2x$ true. This means that every choice of the variable x will result in a false equation. Attempt to solve the equation using the normal method. Describe what your equation looks like and why it makes sense to conclude that all values of x make the equation false.

4.6 Deliberate Practice: Solving Equations for a Variable (Part 2)

Focus on these skills:

- Write the original equation.
- Line up your equal signs.
- Be careful with negative signs and the distributive property.
- Present your work legibly.

Instructions: Solve the equation for the variable.

1 $-8n + 4(1 + 5n) = -6n - 13$

2 $8x + 4(4x - 3) = 4(6x - 4) + 4$

3 $-3(v - 1) + 8(v - 3) = 6v + 7 - 5v$

4 $4m - 40 = 7(-2m + 3)$

5 $-47 + p = -5(8p + 10)$

6 $9(x - 9) - 3 = -4(2x + 4)$

7 $-2(2 + a) + 1 = 3(2 - 3a)$

8 $4(-8y + 5) = -15y - 26$

9 $-7(5 + x) = -56 - 6x$

10 $6m + 2(13m - 5) + 2 = -2(4 - 16m)$

4.7 Closing Ideas

We spent a lot of time in this section discussing how mathematics is presented. One of the main challenges of this is that there are no formal rules that tell you how much or how little writing is required. How much work you show depends on the audience that will be reading the work.

Let's take another look at the example from the worksheet of the very long presentation:

$$\begin{aligned} 7x &= x + 24 && \\ 7x - x &= x + 24 - x && \text{Subtract } x \text{ from both sides} \\ 7x - x &= x - x + 24 && \text{Rearrange the terms} \\ 7x - x &= (x - x) + 24 && \text{Group the terms} \\ (7-1)x &= (1-1)x + 24 && \text{Factor out the } x \\ 6x &= 0x + 24 && \text{Arithmetic} \\ 6x &= 24 && \text{Simplify} \\ \frac{6x}{6} &= \frac{24}{6} && \text{Divide both sides by 6} \\ x &= 4 && \text{Arithmetic} \end{aligned}$$

As was noted in that problem, there is nothing wrong with the algebra in this presentation. And if someone asked you to show all of the work, this is what it would look like. But in practice, we would never do this. Why? Because in the context of solving an equation, it's natural to assume that the reader already understands how to combine like terms. The focus of the problem is on the steps needed to solve the equation, and not the steps required to combine like terms.

A more normal level of presentation for that problem would be something like this:

In some situations, it is customary not to show any of the algebra!

$$\begin{aligned} 7x &= x + 24 && \\ 6x &= 24 && \text{Subtract } x \text{ from both sides} \\ x &= 4 && \text{Divide both sides by 6} \end{aligned}$$

Notice how each step in the presentation is focused on the actual steps of solving the equation, and all the little steps are not shown. This is because the presentation is being tailored to match the goal of the problem.

Reading the instructions of a problem should give you a sense of what the problem is asking you to do, and from that information you can also make decisions about what steps are important and which ones are less so.

4.8 Going Deeper: Inequalities

In this section, we talked about the difference between equations and expressions. But there's another type of mathematical statement that is like an equation, but instead of declaring that two quantities represent the same value, we want one to be greater than or less than the other. These statements are known as inequalities.

Definition 4.5. A mathematical *inequality* is a statement of the form $A > B$, $A < B$, $A \geq B$, or $A \leq B$, where A and B are mathematical expressions. Each statement has an associated interpretation:

- $A > B$ means that A represents a quantity that is greater than the quantity that B represents.
- $A < B$ means that A represents a quantity that is less than or equal to the quantity that B represents.
- $A \geq B$ means that A represents a quantity that is greater than or equal to the quantity that B represents.
- $A \leq B$ means that A represents a quantity that is less than or equal to the quantity that B represents.

In Section 24, we will see how to visualize these relationships geometrically. This geometric framework is particularly useful when negative numbers are involved. But for now, we're going to trust your intuition and experience to interpret the meaning of these words.

We often call the first two symbols *strict* inequalities, emphasizing that we are interested in the more stringent condition. Interestingly, we don't have a formal name for the other two. You will sometimes see them called *non-strict*, *inclusive*, or *weak* inequalities, but there's no consensus term among mathematicians.

Manipulating inequalities is almost identical manipulating equations, though there is one very important distinction. When multiplying or dividing, the axioms only allow for doing this with positive values. You should compare this definition with Definition 1.1.

Definition 4.6. Let a, b, and c be real numbers. The *axioms of inequality* state that

- If $a > b$ then $b < a$.
- If $a > b$, then $a + c > b + c$.
- If $a > b$, then $a - c > b - c$.
- If $a > b$ and $c > 0$, then $ac > bc$.
- If $a > b$ and $c > 0$, then $\frac{a}{c} > \frac{b}{c}$.

The first axiom allows us to state these properties for just the $>$ symbol rather than having to repeat everything with the $<$ symbol. We also avoid having to explicitly discuss that $\geq$ and $\leq$ symbols because those are simply shorthand to represent both a strict inequality and an equality at the same time.

Notice the extra condition that c is positive in the last two axioms!

You might remember from your previous experiences that when you multiply or divide by a negative number, that you're supposed to flip the inequality. But that's not listed here! What's happening is that this property is not an axiom. We can actually show that this property is a logical consequence of the listed properties. We will show how the negative sign appears in a

Rule-based thinking treats everything like a axiom. Every time there's a new situation, there's a new rule to learn. Mathematicians prefer fewer rules because it helps us identify the most important ideas.

simple calculation and let you try to prove the general property on your own.

$$\begin{aligned} a &> b \\ a + ((-a) + (-b)) &> b + ((-a) + (-b)) && \text{Add } (-a) + (-b) \text{ to both sides} \\ -b &> -a && \text{Combine like terms} \\ -a &< -b && \text{The axioms of equality} \end{aligned}$$

Breaking concepts down into basic axioms is like understanding that all atoms are made of protons, neutrons, and electrons. A small set of basic building blocks can create an entire universe of mathematics.

To prove the general case, note that if $c < 0$, then $-c > 0$ so that you can multiply both sides by $-c$ in the first step using the axioms of inequality. From there, you'll need to make a similar manipulation to the one above.

The general proof is a simple calculation, but remember that simple is not always easy. It's normal for students to struggle with this.

We can expand our definition of what it means to solve an equation so that it applies to inequalities. You should compare this definition with Definition 4.4.

Definition 4.7. To *solve* an inequality means to find the value (or values) of the variable (or variables) that make the inequality true. A *solution* to an inequality is a specific value of the variable (or specific values of the variables) that make the inequality true.

With the exception of worrying about the direction of the inequality, solving inequalities is identical to solving equations. In fact, if you go back to any of the problems from previous sections and replace the equal sign with any of the inequality symbols, the algebraic steps will be identical and you would only need to check whether any of the steps involved multiplying or dividing by a negative number. For example, here is "Try It" exercise #2, except using $>$ instead of $=$ in the original problem.

$$\begin{aligned} 5x - 7 &> 7x + 11 \\ 5x &> 7x + 18 && \text{Add 7 to both sides} \\ -2x &> 18 && \text{Subtract } 7x \text{ from both sides} \\ x &< -9 && \text{Divide both sides by } -2 \end{aligned}$$

Compare this to the solution at the end of the section.

Notice that the change happened right at the step where we divided by -2. It doesn't happen before that step, and it doesn't happen on its own line after that step. It also does not happen because the 9 is negative. It happens because the algebraic manipulation called for dividing by a negative number. Furthermore, if you use the improper presentation from the first section of the book, inequalities lead to some rather unusual and nonsensical mathematical statements. We put such a heavy emphasis on presentation in order to avoid many of the errors that can arise from these things.

A key distinction is that solutions to inequalities are usually not a single value, but (usually) an infinite collection of values. For example, the inequality $x > 0$ is true when $x = 1, 2, 3, \ldots$. But we have to remember that we're thinking about all possible values, so we have to include decimals such as $0.1, 0.01, 0.001, 0.0001, \ldots$.

The second collection of values hints at a very unusual feature about numbers. There is no smallest solution to the inequality $x > 0$. Another way of saying this is that there is no smallest positive number. For any positive number you can think of, there's always another that's less than your number but still positive. If you've never really thought about why this happens, it's worth taking some time to think about it.

It turns out that this simple notion is connected to a number of profoundly interesting mathematical ideas. Here are just a few questions that this one idea leads to:

- There are infinitely many numbers between 0 and 1 and infinitely many numbers between 0 and 0.00000001. Since the first gap is bigger than the second gap, does it make sense to say that the infinity of values in the first interval is larger than the infinity of values in the second? (It turns out that these two infinities are the same size, but that isn't the same as saying that *all* infinities are the same!)
- What happens to the value of $\frac{1}{x}$ as we let x take the values $0.1, 0.01, 0.001, 0.0001, \ldots$? It looks like it's getting larger and larger. Does it make sense to say that $\frac{1}{0} = \infty$? (It turns out that it doesn't, but the exact reason this idea fails is a bit subtle.)
- Does it make sense to have a number $0.000\ldots0001$, where there are infinite number of zeros before the 1? (It turns out that such a mathematical object can make sense if you think about it the right way, but it's no longer what we would call a number. Instead, it's what me might call a *surreal* number. And no, this is not a joke.)

What does it even mean for something to come *after* an infinite number of other things?

We won't go into further detail about any of these topics here, as it goes way beyond the scope of this course. But it is interesting to reflect on how some very basic questions can lead to very profound mathematical ideas.

4.9 Solutions to the “Try It” Examples

1

$(3x + 7y) - (4x + 5y) = 3x + 7y - 4x - 5y$	Distributive property
$= 3x - 4x + 7y - 5y$	Rearrange the terms
$= (3x - 4x) + (7y - 5y)$	Group the terms
$= (3 - 4)x + (7 - 5)y$	Factor out the x and the y
$= -x + 2y$	Arithmetic

2

$5x - 7 = 7x + 11$	
$5x = 7x + 18$	Add 7 to both sides
$-2x = 18$	Subtract $7x$ from both sides
$x = -9$	Divide both sides by -2

This is a That: Variables and Substitutions

Learning Objectives:

- ☐ Substitute a number for a variable then simplify or solve.
- ☐ Substitute a variable expression for another variable then simplify or solve.

We have previously seen how we can substitute a number for a variable. This led us to write sentences like "If $x = 3$, then $2x + 1 = 7$."

In math, we often loop back to previous ideas to give ourselves a chance to go deeper.

We are not restricted to just single variable expressions. It is often the case that a problem is modeled using multiple variables. Algebraically, this is just a matter of making sure you use the correct value for the correct variable. But there are some potential pitfalls as the substitutions become more complex.

1 The problem is fundamentally about demonstrating that you can make substitutions and perform the corresponding arithmetic correctly. When making a substitution, it is helpful to be in the habit of putting that substitution inside of parentheses. Sometimes it's necessary, and sometimes it's extraneous. Over time, you will have the experience and insight to see whether it's necessary, but to start off it's best be in the habit of using parentheses all the time.

Here is a presentation for finding the value of $3a - 2b$ when $a = 2$ and $b = 4$:

$$\begin{aligned} 3a - 2b &= 3(2) - 2(4) && \text{Substitute } a = 2 \text{ and } b = 4 \\ &= 6 - 8 && \text{Arithmetic} \\ &= -2 && \text{Arithmetic} \end{aligned}$$

If $a = 2$ and $b = 4$, then $3a - 2b = -2$.

Some people are comfortable doing both arithmetic steps mentally, while others aren't. For this section, we recommend showing your work at about this level to help identify any errors in your mental arithmetic.

Try it: Determine the value of the expression $2x + 5y$ when $x = 3$ and $y = -2$. Show the calculation and write your result as an if-then statement.

A major step in your mathematical thinking is the ability to apply previous results in new settings, and combining old ideas in new ways. We have talked about substituting a value for a variable, and we've talked about solving for a variable. We can combine the two into a new type of problem.

2 Here is an example of combining the two ideas together. Consider the equation $3m + 5n = 30$. This equation has two variables in it. If we declare a value for one of the variables, then we

In this case, no matter what value of m or n is given to you, there will always be a solution. But for some equations, this won't happen.

can solve for the other one. For example, if we are told that $m = 5$, then we have the following:

$$\begin{aligned} 3m + 5n &= 30 && \\ 3(5) + 5n &= 30 && \text{Substitute } m = 5 \\ 15 + 5n &= 30 && \text{Arithmetic} \\ 5n &= 15 && \text{Subtract 15 from both sides} \\ n &= 3 && \text{Divide both sides by 5} \end{aligned}$$

Try it: Solve the equation $3m + 5n = 30$ for m when $n = -2$. Use a complete presentation.

Don't be afraid of fractions.

There are other types of substitutions that we can make. In the definition of a variable (Definition 2.1), we said that variables can also represent mathematical expressions. Fortunately, there is no conceptual difference between the two. It all comes down to the proper execution of algebra.

3 When substituting an expression for a variable, it is important to wrap the expression inside of parentheses. This small detail is extremely important because it represents a container that holds the entire contents of the variable. Failure to do so will often lead to failing to properly apply the distributive property or committing other types of errors.

This is how it looks to substitute $x = 2y - 1$ into the expression $-x + 6$:

$$\begin{aligned} -x + 6 &= -(2y - 1) + 6 && \text{Substitute } x = 2y - 1 \\ &= -2y + 1 + 6 && \text{Distributive property} \\ &= -2y + 7 && \text{Combine like terms} \end{aligned}$$

Try it: Substitute $a = -2b + 3$ into the expression $-2a - 5$ and simplify the result. Use a complete presentation.

4 Conceptually, substituting into an expression is not different from substituting into an equation because an equation is just two expressions that are claimed to represent the same value.

Try it: Substitute $x = 2y + 1$ into the equation $3x - 2y = 7$ and solve for y. Use a complete presentation.

This might remind you of the substitution method of solving linear equations. If you don't know what that is, we'll get to it later.

5.1 Variables and Substitutions - Worksheet 1

1 Determine the value of the expression $3p + 5q$ when $p = -2$ and $q = -1$. Show the calculation and write your result as an if-then statement.

The most common errors on these problems tend to be the simple arithmetic steps. Take your time and be careful!

2 Substitute $b = 3a - 5$ into the expression $2b + 3$ and simplify the result. Use a complete presentation.

3 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

Be aware of this error so that you can avoid making it yourself!

$-x + 5 = -y + 3 + 5$	Substitute $x = y + 3$
$= -y + 8$	Arithmetic

5.2 Variables and Substitutions - Worksheet 2

1 Determine the value of the expression $4a - 3b$ when $a = -3$ and $b = 4$. Show the calculation and write your result as an if-then statement.

2 Substitute $n = -2m + 1$ into the expression $-n + 2$ and simplify the result. Use a complete presentation.

3 Substitute $y = 2x - 3$ into the expression $2x + 3y$ and simplify the result. Use a complete presentation.

5.3 Variables and Substitutions - Worksheet 3

1 Determine the value of the expression $x - y^2$ when $x = 2$ and $y = -3$. Show the calculation and write your result as an if-then statement.

2 Solve the equation $-3a + 4b - 7 = 5$ for a when $b = -2$. Use a complete presentation.

3 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\begin{aligned} -3p + 4q &= 5 & & \\ -3(q-4) + 4q &= 5 & & \text{Substitute } p = q - 4 \\ -3q - 12 + 4q &= 5 & & \text{Distributive property} \\ q - 12 &= 5 & & \text{Combine like terms} \\ q &= 17 & & \text{Add 12 to both sides} \end{aligned}$$

5.4 Variables and Substitutions - Worksheet 4

1 Substitute $s = -2t + 1$ into the equation $2s + 1 = 4$ and solve for the variable. Use a complete presentation.

2 Solve the equation $3m - 2n - 7 = 5$ for n when $m = 4n - 3$. Use a complete presentation.

3 Substitute $x = 3$ into the expression $x^2 - 6x + 9$ and simplify the result. Use a complete presentation.

The value of x must be the same for the entire expression. We do not want x to take two different values at the same time.

5.5 Variables and Substitutions - Worksheet 5

1 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$5x - y = 4$	
$5x - (3x + 4) = 4$	Substitute $y = 3x - 2$
$5x - 3x - 4 = 4$	Distributive property
$2x - 4 = 4$	Combine like terms
$2x = 0$	Subtract 4 from both sides
$x = 0$	Divide both sides by 2

2 Substitute $n = -2m - 3$ into the expression $-3m - 2n + 5$ and simplify the result. Use a complete presentation.

3 Solve the equation $-2a + 3b - 7 = b$ for a when $b = -a + 2$. Use a complete presentation.

5.6 Deliberate Practice: Substitute and Solve

Focus on these skills:

- Write the original equation.
- Use parentheses with the substitution.
- Be careful with negative signs and the distributive property.
- Present your work legibly.

Instructions: Substitute the first equation into the second equation, then solve for the remaining variable.

1. $y = 3x + 4; \quad 2x + 3y = -6$

2. $a = 3b - 2; \quad 4a - 3b = 7$

3. $m = n - 4; \quad -2m + 3m - 5 = -8$

4. $t = -2s + 3; \quad 3s - 2t + 7 = 7$

5. $b = -3a + 1; \quad 4a - b - 4 = 4$

6. $x = -y + 3; \quad -3x - 4y + 6 = 0$

7. $u = 2v - 4; \quad 3u + 4v - 3 = -2u - 5$

8. $r = 3s + 1; \quad -r + 3s - 3 = 4r - 3s + 7$

9. $x = -2y + 3; \quad 2x - y + 4 = x - 3y - 6$

10. $n = -3m - 4; \quad 2m - 3n - 1 = -2m + 4n + 5$

5.7 Closing Ideas

The use of variables to represent other variable expressions is an extremely important skill in mathematics. In your college math course, you will likely see that function notation can lead you to make the exact same type of substitutions. Here is a brief preview of that.

Suppose we define the function $f(x) = x^2 - 2$. To evaluate the function means to substitute for the variable and simplify the result. Here are some examples.

A function is a rule that tells us what to do to the number x.

- $f(3) = 7$ since $(3)^2 - 2 = 7$.
- $f(-2) = 2$ since $(-2)^2 - 2 = 2$

But students start to struggle with this concept as soon as we start substituting objects other than numbers:

- $f(y) = y^2 - 2$
- $f(x-3) = (x-3)^2 - 2$

(In the second case, you might be asked to simplify the expression.)

The actual concept involved is exactly what we've done in this section. We've simply replaced the variable of the function with a variable expression. But students who lack a clear understanding of this concept end up with all sorts of wrong answers. Here are some examples:

- $f(x-3) = (x^2-3) - 2$
- $f(x-3) = x - 3^2 - 2$
- $f(x-3) = (x^2+2) - 3$

These sorts of erroneous manipulations stem from a much deeper place than most students realize. Some students don't even want to try to think through the problem, and so they just guess and hope for the best. And there's some human psychology to this. They feel so accustomed to being wrong in math classes that they've simply given up trying to be right. Their mentality has shifted to "I'm just going to write down something and wait for someone to tell me what the right thing is." And so they just scribble down symbols and hope for the best. As we keep making our way through the different topics, remember that mathematical thinking includes having a clear sense of what you're doing and why you're doing it. Everything is supposed to be logical, and everything is supposed to make sense.

The emotional battle of mathematics is usually the far greater challenge than the intellectual one.

Most of the errors above stem from students simply not taking the time to think carefully. In fact, many students can find their errors after they start to try to explain what they're doing and why they're doing it. In other words, most students know enough to fix their own errors, if they would just take the time to think more carefully about it.

Sometimes, you just need to stop and think.

Success in mathematics is attainable through the consistent effort of careful and thoughtful practice. It's not about being fast or even being smart. Math is a learned skill like every other learned skill, which means that you can learn it by putting in the time and the effort. So do not sell yourself short by refusing to try to learn. You're going to make mistakes, and that's okay. Just keep trying to do a little bit better every time.

Careful and thoughtful practice can be enhanced by seeking academic support mechanisms such as tutoring and office hours. These things exist as a pathway to help students become successful, and you should not be afraid to take advantage of them.

5.8 Going Deeper: Disappearing Variables

In all the equations that we've solved so far, it has always been the case that the last line of work has given us a specific number or expression for the variable of interest. Let's look at solving the equation $3x + 7 = -2$.

$$\begin{aligned} 3x + 7 &= -2 \\ 3x &= -9 && \text{Subtract 7 from both sides} \\ x &= -3 && \text{Divide both sides by 3} \end{aligned}$$

We have been interpreting the last line as telling us that when $x = -3$, the original equation is true. We can even check this by explicit substitution.

$$\begin{aligned} 3x + 7 &= -2 \\ 3(-3) + 7 &\stackrel{?}{=} -2 && \text{Substitute } x = -3 \\ -2 &\stackrel{\checkmark}{=} -2 \end{aligned}$$

We use $\stackrel{?}{=}$ to ask whether the two sides are equal to each other after a substitution. At the end of the calculation, we will replace it with $\stackrel{\checkmark}{=}$ if it is and with $\neq$ if it is not.

But what happens when the variable disappears? Consider the following attempt at solving an equation:

$$\begin{aligned} 3(x + 2) + 5 &= 3x + 7 \\ 3x + 6 + 5 &= 3x + 7 && \text{Distribute} \\ 3x + 11 &= 3x + 7 && \text{Arithmetic} \\ 11 &= 7 && \text{Subtract } 3x \text{ from both sides} \end{aligned}$$

How can we interpret the this? The first thing to observe is that the equation is false. The numbers 11 and 7 are definitely not the same. What this is telling us is that there are no values of the variable that make the equation true. The equation will always be false, no matter what value we choose the variable to be. A concise way of saying this is to say that there are no solutions to the equation. We can pick some values of x and check this.

We could also say that the solution set is $\emptyset$.

$$\underline{x = 1}$$
$$\begin{aligned} 3(x + 2) + 5 &= 3x + 7 \\ 3(1 + 2) + 5 &\stackrel{?}{=} 3(1) + 7 \\ 14 &\neq 10 \end{aligned}$$

$$\underline{x = 2}$$
$$\begin{aligned} 3(x + 2) + 5 &= 3x + 7 \\ 3(2 + 2) + 5 &\stackrel{?}{=} 3(2) + 7 \\ 17 &\neq 13 \end{aligned}$$

$$\underline{x = 3}$$
$$\begin{aligned} 3(x + 2) + 5 &= 3x + 7 \\ 3(3 + 2) + 5 &\stackrel{?}{=} 3(3) + 7 \\ 20 &\neq 16 \end{aligned}$$

$$\underline{x = 4}$$
$$\begin{aligned} 3(x + 2) + 5 &= 3x + 7 \\ 3(4 + 2) + 5 &\stackrel{?}{=} 3(4) + 7 \\ 23 &\neq 19 \end{aligned}$$

Notice that the values we're getting in these calculations are not the 11 and 7 that we got above. Notice how we subtracted $3x$ from both sides of the equation in the last step. What would happen to our values if we also did that here?

Here is something else that may happen:

$$\begin{aligned} 2(2x-1)+3 &= 4(x+1)-3 & & \\ 4x-2+3 &= 4x+4-3 & & \text{Distribute} \\ 4x+1 &= 4x+1 & & \text{Arithmetic} \\ 1 &= 1 & & \text{Subtract } 4x \text{ from both sides} \end{aligned}$$

Once again, the variable disappeared. But this time, we have a true equation. What this means is that regardless of the value of the variable, the equation will always be true. A shorter way of saying that is that x can be any real number. We will check a few examples to demonstrate this.

We could also say that the solution set is $\mathbb{R}$.

$$\begin{gathered} \underline{x=1} \\ 2(2x-1)+3 = 4(x+1)-3 \\ 2(2(1)-1)+3 \stackrel{?}{=} 4(1+1)-3 \\ 5 \stackrel{\checkmark}{=} 5 \end{gathered} \qquad \begin{gathered} \underline{x=2} \\ 2(2x-1)+3 = 4(x+1)-3 \\ 2(2(2)-1)+3 \stackrel{?}{=} 4(2+1)-3 \\ 9 \stackrel{\checkmark}{=} 9 \end{gathered}$$

$$\begin{gathered} \underline{x=3} \\ 2(2x-1)+3 = 4(x+1)-3 \\ 2(2(3)-1)+3 \stackrel{?}{=} 4(3+1)-3 \\ 13 \stackrel{\checkmark}{=} 13 \end{gathered} \qquad \begin{gathered} \underline{x=4} \\ 2(2x-1)+3 = 4(x+1)-3 \\ 2(2(4)-1)+3 \stackrel{?}{=} 4(4+1)-3 \\ 17 \stackrel{\checkmark}{=} 17 \end{gathered}$$

These calculations don't *prove* that we will always get an equality. There are an infinite number of other numbers that we haven't checked yet! How can we know that there isn't some special number out there that breaks this equality?

As we continue with various types of algebraic calculations and manipulations, there will be other times when a variable will be missing from an equation or an expression. For example, consider the following: Substitute $x = 4$ and $y = 2$ into the expression $2x - 7$. The substitution into the x term is clear, but what does it mean to substitute for y when there is no y variable?

The insight comes from looking at our calculations above. What caused the variable terms to disappear? It's both obvious and subtle at the same time. In the last step, we subtracted off a certain quantity that caused the variable terms to cancel out. Let's break that down more carefully:

$$\begin{aligned} 4x+1 &= 4x+1 & & \\ (4x+1)-4x &= (4x+1)-4x & & \text{Subtract } 4x \text{ from both sides} \\ 4x-4x+1 &= 4x-4x+1 & & \text{Rearrange the terms} \\ (4-4)x+1 &= (4-4)x+1 & & \text{Group and factor out the } x \\ 0x+1 &= 0x+1 & & \text{Arithmetic} \\ 0+1 &= 0+1 & & \text{Arithmetic} \\ 1 &= 1 & & \text{Arithmetic} \end{aligned}$$

We are expanding the very last step of the $2(2x-1)+3 = 4(x+1)-3$ calculation.

The last two arithmetic steps are what cause the variable term to go away. We're using the fact that zero multiplied by any number is zero and that zero added to any number does not change the value. And by these observations, we are explaining why it makes sense for us to simplify the expression so that the variable is not shown. But this does not say that the variable doesn't exist anymore. In some sense, the $0x$ term is still there. We're just not bothering to write it down because it doesn't have any impact on the quantity being expressed.

And that brings us back to the two-variable substitution. Even though we're only writing $2x - 7$, you should really be thinking about it as something like $2x + 0y - 7$. We say that the expression $2x - 7$ is *independent of the variable* y, and that simply means that changing the value of y has no impact on the value of the expression.

You can also think of it as a $0y^2$ term or even terms like $0xy$. As long as the coefficient of the monomial is zero, that term can be dropped.

The first place that students run into this is when we define functions. For example, let's say we define the function $f(x) = 4$. What is $f(2)$? We know that this means we're supposed to plug in 2 for x, but there are no x variables anywhere. The expression is independent of x and so $f(x) = 4$ regardless of what value of x is chosen.

This may seem like a small point, but the concept of functions or other expressions being independent of certain variables comes up regularly in both pure and applied mathematical settings. So it's helpful to understand this idea sooner rather than later.

5.9 Solutions to the "Try It" Examples

1

$$\begin{aligned} 2x + 5y &= 2(3) + 5(-2) && \text{Substitute } x = 3 \text{ and } y = -2 \\ &= 6 + (-10) && \text{Arithmetic} \\ &= -4 && \text{Arithmetic} \end{aligned}$$

If $x = 3$ and $y = -2$, then $2x + 5y = -4$.

2

$$\begin{aligned} 3m + 5n &= 30 && \\ 3m + 5(-2) &= 30 && \text{Substitute } n = -2 \\ 3m + (-10) &= 30 && \text{Arithmetic} \\ 3m &= 40 && \text{Add 10 to both sides} \\ m &= \frac{40}{3} && \text{Divide both sides by 3} \end{aligned}$$

3

$$\begin{aligned} -2a - 5 &= -2(-2b + 3) - 5 && \text{Substitute } a = -2b + 3 \\ &= 4b - 6 - 5 && \text{Distributive property} \\ &= 4b - 11 && \text{Combine like terms} \end{aligned}$$

4

$$\begin{aligned} 3x - 2y &= 7 && \\ 3(2y + 1) - 2y &= 7 && \text{Substitute } x = 2y + 1 \\ 6y + 3 - 2y &= 7 && \text{Distributive property} \\ 4y + 3 &= 7 && \text{Combine like terms} \\ 4y &= 4 && \text{Subtract 6 from both sides} \\ y &= 1 && \text{Divide both sides by 4} \end{aligned}$$

Don't Memorize These Formulas: The Properties of Exponents

Learning Objectives:

- ☐ Interpret the meaning of integer exponents.
- ☐ Explain the derivation of the product and power rules for exponents.
- ☐ Correctly execute calculations involving the product and power rules for exponents.

In an earlier section, we had worked with monomial and polynomial expressions that involved exponents. We implicitly assumed that you were familiar with the notation. We're going to formally define exponents here for the sake of completeness.

Definition 6.1. A *positive integer exponent* represents repeated multiplication. The expression a^n represents the quantity a multiplied by itself n times. In other words,

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}.$$

The positive integers are the numbers 1, 2, 3, 4, 5, and so forth.

This definition allows us to write out exponent expressions "the long way" by explicitly writing out the terms in the product:

$$\begin{aligned} x^1 &= x \\ x^2 &= x \cdot x \\ x^3 &= x \cdot x \cdot x \\ &\vdots \end{aligned}$$

1 This definition has a consequence that can be seen through some examples. Consider the following products:

$$x^3 \cdot x^4 = \underbrace{\underbrace{(x \cdot x \cdot x)}_{3\text{ times}} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{4\text{ times}}}_{7\text{ times}} = x^7 \qquad x^4 \cdot x^1 = \underbrace{\underbrace{(x \cdot x \cdot x \cdot x)}_{4\text{ times}} \cdot \underbrace{(x)}_{1\text{ time}}}_{5\text{ times}} = x^5$$

"But the equal signs aren't lined up!" Relax. It's not a rule. It's just an organizational tool. Readability comes first.

Try it: Based on the pattern observed above, what would you say $x^m \cdot x^n$ should be equal to? Write out an explanation in both words and an equation (similar to the ones above) that explains why the pattern exists.

Being able to explain your ideas in sentences helps you to understand your ideas more deeply.

2 There is a second consequence of the definition of exponents that can also been seen through some examples:

$$(x^4)^2 = \underbrace{\underbrace{(x \cdot x \cdot x \cdot x)}_{\text{4 times}} \cdot \underbrace{(x \cdot x \cdot x \cdot x)}_{\text{4 times}}}_{\text{2 groups of 4 times}} = x^8 \qquad (x^2)^3 = \underbrace{\underbrace{(x \cdot x)}_{\text{2 times}} \cdot \underbrace{(x \cdot x)}_{\text{2 times}} \cdot \underbrace{(x \cdot x)}_{\text{2 times}}}_{\text{3 groups of 2 times}} = x^6$$

Try it: Based on the pattern observed above, what would you say $(x^m)^n$ should be equal to? Write out an explanation in both words and an equation (similar to the ones above) that explains why the pattern exists.

These two results are important enough that they have names.

Theorem 6.2.

- *The Product Rule for Exponents:* $x^m \cdot x^n = x^{m+n}$
- *The Power Rule for Exponents:* $(x^m)^n = x^{mn}$

These may be "rules" but they are not arbitrary. If you ever get confused about whether to add or multiply the exponents, just think about the explanations you just wrote out.

Did you notice how Definition 6.1 emphasizes that it only applies to positive integers? This is because the definition requires us to count the number of terms in the product. But if that's the case, what should x^0 be? Should it be 0? Should it be 1? What about negative powers of x? Let's see if we can come up with a sensible pattern. Consider the following diagram:

Math is all about finding patterns. When we guess at something, it should be a guess based on a pattern, and not just a random guess.

At each step:
Add 1 to the exponent
Multiply the right side by x

$$\uparrow \quad \begin{aligned} x^4 &= x \cdot x \cdot x \cdot x \\ x^3 &= x \cdot x \cdot x \\ x^2 &= x \cdot x \\ x^1 &= x \end{aligned}$$

This gives us an accurate picture of the pattern of exponents. Starting from any equation, we can get the next one by adding 1 to the exponent and multiplying the right side by another x. We're now going to try to turn this around and go backwards:

At each step:
Subtract 1 from the exponent
Divide the right side by x

$$\downarrow \quad \begin{aligned} x^4 &= x \cdot x \cdot x \cdot x \\ x^3 &= x \cdot x \cdot x \\ x^2 &= x \cdot x \\ x^1 &= x \end{aligned}$$

So all we need to do is continue the pattern.

Philosophical question: Did we create this pattern, or was it already there and we're just discovering it?

At each step:
Subtract 1 from the exponent
Divide the right side by x

$$\downarrow \quad \begin{aligned} x^1 &= x \\ x^0 &= 1 \\ x^{-1} &= \frac{1}{x} \\ x^{-2} &= \frac{1}{x \cdot x} = \frac{1}{x^2} \end{aligned}$$

These ideas lead us to our next definition, which completes the definitions of exponents for the remaining integers:

Definition 6.3. We define the following notation for $x \neq 0$:

- $x^0 = 1$
- $x^{-n} = \frac{1}{x^n}$

The second equation is true for all negative exponents, even when it's not an integer. But for now, we're going to avoid the fractional and real number exponents.

The condition is worthy of a closer look. The challenge that arises is that the pattern that we had fails when $x = 0$. The reason is that we cannot divide by zero, so inverting the multiplication and turning it into division simply fails.

It's interesting to think about why division by zero is not allowed. It's related to the actual concept of what it means to divide two numbers, but we'll talk more about that one later.

There are two other formulas that we can get by combining these properties.

Theorem 6.4.

- $\left(\frac{1}{x}\right)^{-n} = x^n$
- $\frac{1}{x^{-n}} = x^n$

We will see where these formulas come from in the worksheets.

3 As it turns out, the patterns that were developed above can also be applied when the exponents are not positive integers. Consider the following example:

$$\begin{aligned}
x^5 \cdot x^{-3} &= x^5 \cdot \frac{1}{x^3} && \text{Definition of negative exponents} \\
&= \frac{x^5}{x^3} && \text{Multiply fractions} \\
&= \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} && \text{Definition of exponents} \\
&= \frac{\cancel{x \cdot x \cdot x} \cdot x \cdot x}{\cancel{x \cdot x \cdot x}} && \text{Reduce the fraction} \\
&= x^2 && \text{Simplify}
\end{aligned}$$

When we cancel terms in a fraction, they reduce to a 1, so this is $\frac{1 \cdot x^2}{1}$.

If we apply pattern from above, we get the same result:

$$\begin{aligned}
x^5 \cdot x^{-3} &= x^{5+(-3)} && \text{Product rule for exponents} \\
&= x^2 && \text{Arithmetic}
\end{aligned}$$

Try it: Calculate $x^2 \cdot x^{-5}$ using both types of presentations above. Your final result should be of the form x^n for some integer n.

4 The power rule for exponents also works when the exponents are zero or negative. You have all the tools you need to demonstrate this for yourself.

Try it: Calculate $\left(x^{-2}\right)^3$ using a presentation that shows all of the individual steps. Then verify that the power rule gives the same result.

6.1 Properties of Exponents - Worksheet 1

1 Calculate $t^2 \cdot t^4$ using a presentation that shows all of the individual steps. Then verify that the product rule gives the same result.

Why write it out when there's a formula? It's important to really understand the formula and not just use it. Otherwise, you might apply the wrong formula at the wrong time and not understand why things are off.

2 Calculate $\left(a^3\right)^4$ using a presentation that shows all of the individual steps. Then verify that the power rule gives the same result.

3 Calculate $y^{-3} \cdot y^5$ using a presentation that shows all of the individual steps. Then verify that the product rule gives the same result.

6.2 Properties of Exponents - Worksheet 2

1 Calculate $x^{-2} \cdot x^{-3}$ using a presentation that shows all of the individual steps. Then verify that the product rule gives the same result.

2 Calculate $\left(x^{-1}\right)^{-n}$ using the power rule. Then rewrite the part of the expression inside the parentheses using the definition of negative exponents. In order for the math to be consistent, the two results should be equal. Explain how this verifies the first formula in Theorem 6.4.

Mathematicians like their formulas to be as consistent as possible.

3 Start from the equation $x^{-n} = \frac{1}{x^n}$ and take the reciprocal of both sides of the equation. Explain how this verifies the second formula in Theorem 6.4.

Recall that the reciprocal of the fraction $\frac{a}{b}$ is $\frac{b}{a}$.

6.3 Properties of Exponents - Worksheet 3

1 Calculate $x^2 \cdot x^{-5}$ using a presentation that shows all of the individual steps. Then verify that the product rule gives the same result. Give your final answer in the form x^n for some number n.

2 Calculate $x^{2n} \cdot x^{3n}$ using the product rule. Explain the logic of your result in complete sentences.

Remember that variables are just numbers in disguise. What would you do for this problem if the exponents were just numbers?

3 Consider the following presentation:

$$\begin{aligned} x^3 \cdot x^{-3} &= x^3 \cdot \frac{1}{x^3} && \text{Definition of negative exponents} \\ &= \frac{x^3}{x^3} && \text{Multiply fractions} \\ &= \frac{x \cdot x \cdot x}{x \cdot x \cdot x} && \text{Definition of exponents} \\ &= \frac{\cancel{x \cdot x \cdot x}}{\cancel{x \cdot x \cdot x}} && \text{Reduce the fraction} \\ &= 0 \end{aligned}$$

Identify and explain the error. What would you suggest as a way for students to avoid this mistake?

6.4 Properties of Exponents - Worksheet 4

1 Calculate $\left(x^3\right)^{-4}$ using a presentation that shows all of the individual steps. Then verify that the power rule gives the same result. Give your final answer in the form x^n for some number n.

2 Calculate $\left(x^{2m}\right)^{3n}$ using the power rule. Explain the logic of your result in complete sentences.

3 Calculate $x^4 \cdot x^{-4}$ using a presentation that shows all of the individual steps. Then verify that the product rule gives the same result.

Write your answer using the simplest notation possible.

6.5 Properties of Exponents - Worksheet 5

1 Calculate $\left(x^{-3}\right)^{-4}$ using a presentation that shows all of the individual steps. Then verify that the power rule gives the same result. Give your final answer in the form x^n for some number n.

You may want to review Theorem 6.4 for showing all of the steps.

2 Calculate $x^n \cdot x^m \cdot x^p$. Explain the logic of your result.

Mathematicians look for ways to generalize our results so that the ideas we have can be applied to more situations.

3 Calculate $((x^n)^m)^p$. Explain the logic of your result.

6.6 Deliberate Practice: Exponents

Focus on these skills:

- Write the original expression.
- Imagine writing out the various groupings of the variables to reinforce the specific concepts that connect to the formulas.
- Pay close attention to the interplay between negative exponents and fractions.
- Present your work legibly.

Instructions: Simplify the expression, expressing your final answer without fractions.

1. $x^3 \cdot x^6 \cdot x^{-5}$

2. $\left(a^{-4}\right)^3 \cdot a^2$

3. $m^3 \cdot m^2 \cdot m^{-5}$

4. $s^2 \cdot \left(s^3\right)^{-1}$

5. $y^{-3} \cdot y^6 \cdot y^{-4}$

6. $(n^3 \cdot n^2)^{-2} \cdot n^4$

7. $\dfrac{b^3 \cdot b^5}{b^5}$

8. $\dfrac{p^4 \cdot p^3}{\left(p^2\right)^3}$

9. $\dfrac{v^{-2} \cdot \left(v^3\right)^2}{v^3 \cdot v^{-1}}$

10. $\dfrac{\left(z^{-3}\right)^{-2} \cdot z^4}{z^2 \cdot \left(z^{-3}\right)^{-2}}$

6.7 Closing Ideas

There is a balance between writing out all the steps and just using formulas. Just as with other algebraic steps (such as combining like terms), eventually the expectation is that both the person writing things out and the person reading it will be comfortable enough that all of the individual steps will not need to be written out.

But even as you get more experience, there will come moments when you can't quite remember the formula. When this happens, there is a mantra you can use to help jog your memory: "When in doubt, write it out!" What this means is that you can always fall back on the basic ideas to recover these formulas.

Please do not simply guess and hope you guessed correctly! Take responsibility for your knowledge and make the intellectual effort to do the best you can!

Remember that exponents, in their most basic form, are just a shorthand notation for repeated multiplication. After that, it's just logic.

The Product Rule for Exponents:

$$x^m \cdot x^n = \underbrace{\underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \cdot \underbrace{(x \cdot x \cdots x)}_{n \text{ times}}}_{m+n \text{ times}} = x^{m+n}$$

The Power Rule for Exponents:

$$(x^m)^n = \underbrace{\underbrace{(x \cdot x \cdots x)}_{\text{m times}} \cdot \underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \cdots \underbrace{(x \cdot x \cdots x)}_{m \text{ times}}}_{n \text{ groups of } m \text{ times}} = x^{mn}$$

6.8 Going Deeper: Radicals and Fractional Exponents

At some point in the past, you might have remembered learning that $x^{\frac{1}{2}} = \sqrt{x}$. Most likely, you learned this as another rule. However, you have enough knowledge at this point to learn why this is the most (and perhaps only) sensible thing that the fractional exponent could represent.

Before we take a look at fractional exponents, we will do a quick review of radicals. The first radical that students encounter is the square root. The symbol $\sqrt{x}$ represents the non-negative number that has the property that $(\sqrt{x})^2 = x$. Certain integers are considered to be *perfect squares* because their square roots are themselves integers. For example, we have $\sqrt{4} = 2$ (since $2^2 = 4$) and $\sqrt{25} = 5$ (since $5^2 = 25$). Square roots of other numbers exist, and we can either represent the exact values symbolically (such as $\sqrt{2}$ being the number such that $\left(\sqrt{2}\right)^2 = 2$) or we can use a decimal approximation ($\sqrt{2} \approx 1.41421$ because $1.41421^2 = 1.9999899241 \approx 2$).

An important feature of square roots is that we always take the value to be non-negative. When we think about numbers that have the property that squaring them gives you the value 4, we quickly realize that there are two possible values: -2 and 2. This creates a potential ambiguity in our notation, because $\sqrt{4}$ could theoretically be one of two values. However, mathematicians have adopted the convenient convention that the square root function is always non-negative.

This framework clashes with what some students have learned in their previous algebra experiences. Some students have been taught that $\sqrt{4} = \pm 2$. Unfortunately, this is not a proper understanding of the symbols. When we write $\sqrt{4}$, the only value this can represent is 2, and it never represents the value -2. The distinction comes down to understanding what question is being asked:

- What number does $\sqrt{4}$ represent? The value of $\sqrt{4}$ is 2, so that $\sqrt{4} = 2$.
- What numbers x have the property that $x^2 = 4$? The value x can be either -2 or 2, which we can denote as $x = \pm 2$.

In the first case, we're given a specific mathematical expression and are asked to determine its value. In the second case, we are given a mathematical equation and are asked to solve it. These are two different questions, which is why we end up with two different answers.

This is a callback to Section 4.

The reason that there are two solutions to $x^2 = 4$ is because squaring a negative number results in a positive number. In fact, raising a negative number to any even power leads to a positive result. But a negative number raised to an odd power remains odd. This observation is helpful for defining higher roots.

Do you know why a negative multiplied by a negative gives a positive?

> **Definition 6.5.** The *n-th root of x*, denoted $\sqrt[n]{x}$ is the number that has property $\left(\sqrt[n]{x}\right)^n = x$. If n is even, we pick this value to be a nonnegative number (by convention). If $n = 2$, we often omit the n in the notation, so that $\sqrt{x} = \sqrt[2]{x}$.

We can take odd roots of both positive and negative numbers, so that $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$. However, we can only take even roots of positive numbers and the roots are only ever positive values. So $\sqrt[4]{81} = 3$ even though both $3^4 = 81$ and $(-3)^4 = 81$. And it turns out that $\sqrt[4]{-81}$ doesn't exist because it's impossible for a number raised to the fourth power to be negative.

Actually, $\sqrt[4]{-81}$ does exist, but we would first have to create or discover some more numbers to describe it.

This takes us back to the concept of fractional exponents. The key is to think about them in terms of properties. If we expect the properties of exponents to be consistent, then we must have the following:

$$\left(x^{\frac{1}{2}}\right)^2 = x^{\frac{1}{2}\cdot 2} = x^1 = x$$

The first equality is from the power rule for exponents.

This means that whatever $x^{\frac{1}{2}}$ is, it really ought to have the property that when you square it, you get x. And we have already seen that this is the definition of $\sqrt{x}$, which tells us that $x^{\frac{1}{2}} = \sqrt{x}$.

We can use the exact same logic to determine the value of $x^{\frac{1}{3}}$. Since $\left(x^{\frac{1}{3}}\right)^3 = x$, we see that $x^{\frac{1}{3}} = \sqrt[3]{x}$. In fact, we can see that $x^{\frac{1}{n}} = \sqrt[n]{x}$.

What can we say about $x^{\frac{m}{n}}$? Once again, we're going to look at the properties of exponents. Notice that

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$$

and

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}.$$

These calculations give us the ability to evaluate every fractional exponent.

Definition 6.6. For any integer m and positive integer n, we define $x^{\frac{m}{n}}$ to be $\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$.

A key observation about this definition is that we have set it up so that it would be consistent with the product rule and power rule (Theorem 6.2). It is also consistent with negative exponents (Theorem 6.4). This consistency means that there are no new "rules" to learn. This is how mathematicians like to generalize results. It would be very confusing if we had one set of rules that only worked with integer exponents, and then an entirely different set of rules for working with fractional exponents. It makes much more sense to work towards consistent notation and consistent structures.

6.9 Solutions to the "Try It" Examples

1

$$x^m \cdot x^n = \underbrace{\underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \cdot \underbrace{(x \cdot x \cdots x)}_{n \text{ times}}}_{m+n \text{ times}} = x^{m+n}$$

Since x^m means to multiply x by itself m times, and x^n means to multiply by itself n times, if you do the first then the second, you've multiplied x by itself $m+n$ times in total.

2

$$(x^m)^n = \underbrace{\underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \cdot \underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \cdots \underbrace{(x \cdot x \cdots x)}_{m \text{ times}}}_{n \text{ groups of } m \text{ times}} = x^{mn}$$

The product of x multiplied by itself n times is multiplied by itself m times, giving you m groups of x multiplied by itself n times, for a total of mn times that x has been multiplied by itself.

3

$$\begin{aligned}
x^2 \cdot x^{-5} &= x^2 \cdot \frac{1}{x^5} && \text{Definition of negative exponents} \\
&= \frac{x^2}{x^5} && \text{Multiply fractions} \\
&= \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} && \text{Definition of exponents} \\
&= \frac{\cancel{x \cdot x}}{\cancel{x \cdot x} \cdot x \cdot x \cdot x} && \text{Reduce the fraction} \\
&= \frac{1}{x^3} && \text{Simplify} \\
&= x^{-3} && \text{Definition of negative exponents}
\end{aligned}$$

$$\begin{aligned}
x^2 \cdot x^{-5} &= x^{2+(-5)} && \text{Product rule for exponents} \\
&= x^{-3} && \text{Arithmetic}
\end{aligned}$$

4

$$\begin{aligned}
\left(x^{-2}\right)^3 &= x^2 \cdot x^2 \cdot x^2 && \text{Definition of exponents} \\
&= \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} && \text{Definition of negative exponents} \\
&= \frac{1}{x \cdot x} \cdot \frac{1}{x \cdot x} \cdot \frac{1}{x \cdot x} && \text{Definition of exponents} \\
&= \frac{1}{x \cdot x \cdot x \cdot x \cdot x \cdot x} && \text{Multiply fractions} \\
&= \frac{1}{x^6} && \text{Simplify} \\
&= x^{-6} && \text{Definition of negative exponents}
\end{aligned}$$

$$\begin{aligned}
\left(x^{-2}\right)^3 &= x^{-2 \cdot 3} && \text{Power rule for exponents} \\
&= x^{-6} && \text{Arithmetic}
\end{aligned}$$

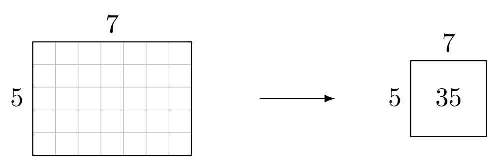
Organization Will Set You Free: Products of Polynomials

Learning Objectives:

- ☐ Multiply monomials.
- ☐ Interpret multiplication as counting boxes in a grid.
- ☐ Use the grid method to multiply polynomials.
- ☐ Understand the distinction between scratch work and formal presentation for polynomial multiplication problems.

Multiplication can be represented in several different ways. For this section, we're going to focus on the idea of multiplication as the area of a rectangle. The area of a rectangle with length ℓ and width w is given by $A = \ell w$. This can also be seen by drawing out the grid and counting the squares, though it's a waste of time to do it in practice. So we will often represent this symbolically by simply using a box to represent the idea of the calculation.

We'll take a deeper look at multiplication in a later section.



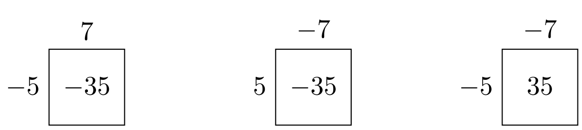
One of the advantages of reducing the idea to just a representation is that it creates a space for us to work with abstract ideas. Once we stop focusing on specific numbers, we can start to use symbols to represent the idea of creating a grid. Consider the following examples:

Abstraction is about focusing on the ideas and not the specific instances. It helps us to see patterns instead of just numbers.

We may not know the particular values of the variables ℓ and w, but we see that if we knew what those values were, then we would be able to draw out the grid to count out the number of squares, and that the number of squares would correspond with the expression inside the box.

We can push this even one step further and think about products that don't even have a proper physical representation:

These pictures can "make sense" but requires you to expand your thinking of area to allow for negative distances and negative areas. What would those even mean and how would we represent them? Those are interesting questions to ponder.



1 Before we can put this to work for polynomials, we first need to focus a bit on multiplying monomials. There were a few problems in the worksheets that hinted at how this works, but we'll formally practice some of those manipulations here.

When multiplying monomials, the result is actually just a single product involving a bunch of different factors. There is usually a number part (from the coefficients) and then there is usually some combination of variables. And all we need to do is properly account for them.

$$\begin{aligned} 5x^2y \cdot 2x^2y^2 &= (5 \cdot x \cdot x \cdot y) \cdot (2 \cdot x \cdot x \cdot y \cdot y) && \text{Definition of exponents} \\ &= 5 \cdot 2 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y && \text{Rearranging the factors} \\ &= 10x^4y^3 && \text{Definition of exponents} \end{aligned}$$

When in doubt, write it out!

This can also be done using the properties of exponents:

$$\begin{aligned} 5x^2y \cdot 2x^2y^2 &= 5 \cdot 2 \cdot x^{2+2} \cdot y^{1+2} && \text{Properties of exponents} \\ &= 10x^4y^3 && \text{Arithmetic} \end{aligned}$$

The amount of explaining you need to do depends on the context of the problem.

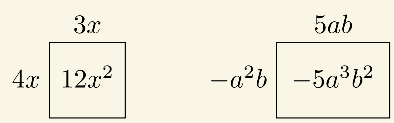
But in the long run, it will be important to be able to do this as just one step.

$$5x^2y \cdot 2x^2y^2 = 10x^4y^3 \qquad \text{Multiplying monomials}$$

The goal is that you would be comfortable and error-free when working at this level.

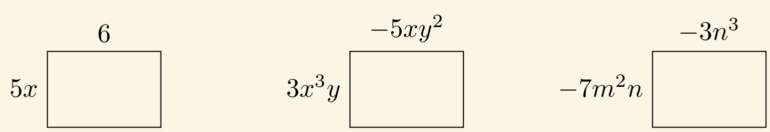
Try it: Compute $-3a^3b^2 \cdot 6ab^3$. Use a presentation that matches each of the three examples above.

2 We can use this framework to set up products using the grid representation that was introduced above.

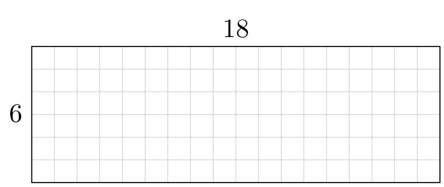


Try it: Complete the products in the following boxes.

See if you can do these products mentally.

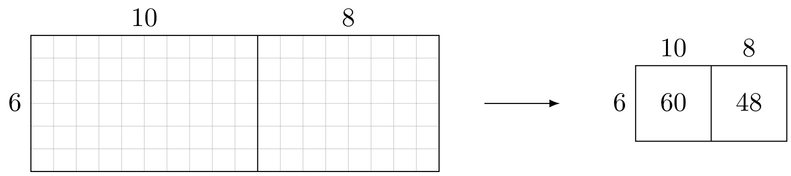


We will now start thinking about how to extend this idea to products of polynomials. Just as before, we will start with concrete values before looking at abstract ideas. Let's say that we wanted to calculate $18 \cdot 6$. We can draw out the picture, but there are a whole lot of squares to count.



Nobody really wants to count those squares, which is why we came up with organizational tools to help us figure out how many there are without direct counting.

So we might start to think about how we can organize this in a more sensible manner. With a little bit of thinking, you might realize that breaking up the 18 into $10+8$ might be helpful. Notice that this doesn't change the number of boxes. However, it does rearrange the information in a way that's more useful to us because we can work with more familiar multiplication calculations.



This is actually a demonstration of the distributive property of multiplication over addition!

3 We can use the same concept for polynomials. For example, we could write the product $(x+2)(x-4)$ using the following grid:

	x	2
x	x^2	$2x$
-4	$-4x$	-8

The grid itself is just a representation of the calculation. We would still need to use formal mathematical writing to present the calculation.

$$\begin{aligned} (x+2)(x-4) &= x^2+2x-4x-8 && \text{Distributive Property} \\ &= x^2-2x-8 && \text{Combining like terms} \end{aligned}$$

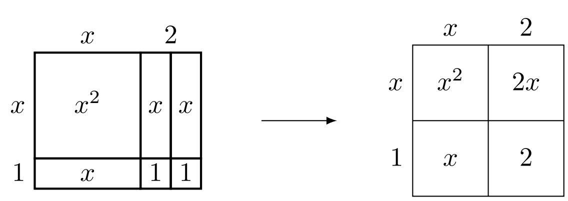
The grid should be understood as scratch work. Scratch work is similar to an outline for an essay. It's important and helpful for keeping yourself organized, but it's not part of the final product. In the end, there must always be enough information in the final presentation so that other people reading your work can know what you did.

The distinction between scratch work and the final presentation is important. Scratch work can be as neat or as messy as you want it to be, but your final presentation should be written up very cleanly.

Try it: Calculate $(2x-3)(x+4)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

There are times when it's helpful to make variables a particular length so that they're different from numbers. For example, we could physically represent the product $(x+2)(x+1)$ using the following diagram.

These are sometimes called "algebra tiles."



We will be using this representation to help us to think about factoring in the next section.

You are probably familiar with FOIL (First-Outer-Inner-Last) as a way to do this product. One of the downfalls to FOIL is that students end up confused when there are more than two terms in the parentheses. Many are so trapped by FOIL that they try to force it to happen even in problems that do not call for it. The basic problem is that students do not have an organized sense of what FOIL is supposed to accomplish, and so they treat it like a rule to be blindly followed.

In math (and in life) it's often unhealthy to blindly follow when someone tells you to do something.

However, this grid method extends very naturally regardless of the number of terms in the parentheses. And it's completely built around the basic idea of thinking of multiplication as an area. Having the right organizational scheme frees you from having to memorize more and more rules.

4

Try it: Calculate $(x^2 + 3x - 4)(x^2 - 5x + 2)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

There is no example for you to follow this time. You're going to have to trust yourself. And even if you're wrong, you can always reach out to your instructor for help. The answer is $x^4 - 2x^3 - 17x^2 + 26x - 8$.

7.1 Products of Polynomials - Worksheet 1

1 Compute the product $4m^2n^3 \cdot 5m^3n$ using three different presentations as demonstrated in this section.

The reason for doing this all three ways is to practice thinking through the logic. In the end, the goal will be that you can do this mentally, but you need to also be able to think through the individual steps correctly.

2 Complete the products in the following boxes.

	8
$3x$	

	$-3ab$
$4a^2b^2$	

	$-2p^2q$
$-6p^3q^2$	

Doing these products mentally is all about practicing keeping yourself organized. Work through the products in a systematic way.

3 Complete the products in the following grids, then write up your results using a complete presentation.

	x	4
5		

	6
$-x$	
3	

7.2 Products of Polynomials - Worksheet 2

1 Compute the product $-3x^4y^3 \cdot 8x^3y$ using three different presentations as demonstrated in this section.

2 Complete the products in the following boxes.

	-4
$2a$	

	$5m^2n$
$-6mn$	

	$7x$
$-3x^4y^5$	

Some students get confused by products like the last one. But the $7x$ term has no y in it! What do you think that means when it comes to taking the product?

3 Calculate $(x + 5)(x + 8)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

7.3 Products of Polynomials - Worksheet 3

1 Complete the products in the following grids, then write up your results using a complete presentation.

	$3x$	$-5y$
$2y$		

	$4a$
$3a$	
-2	

2 Calculate $4(2x+3)$ using a grid. Then write up your result as an equation.

Your final presentation for this will be

$$4(2x+3) = \text{(Result)}.$$

3 Calculate $(x+3)(x-2)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

7.4 Products of Polynomials - Worksheet 4

1 Calculate $4a(-3a+2)$ using a grid. Then write up your result as an equation.

2 Calculate $(2x+4)(x-3)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

3 Calculate $(2a-3b+4)(a-1)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

7.5 Products of Polynomials - Worksheet 5

1 Calculate $(2x+5)(3x-1)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

2 Calculate $(x^2-2x+3)(x-4)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

We generally prefer that the terms of a polynomial are written so that the degree is decreasing. This helps everyone be consistent with each other.

3 Calculate $(x^2+4x-1)(2x^2-3x-2)$ using a grid. Write up your result using a complete presentation, being sure to simplify by combining like terms.

7.6 Deliberate Practice: The Distributive Property

Focus on these skills:

- Write the original expression.
- Visualize or draw out the multiplication grids.
- Simplify by combining like terms.
- Present your work legibly.

Instructions: Calculate the product.

1. $3x(x^2 - 4)$
2. $2x^2(4x^2 - 3x + 6)$
3. $(x + 3)(x - 2)$
4. $(x - 2)(x + 4)$
5. $(2x + 3)(x - 1)$
6. $(2x - 3)(-3x - 4)$
7. $(x^2 + 4)(2x - 5)$
8. $(2x^2 - 3x + 4)(x + 5)$
9. $(x^2 - 2x + 5)(2x - 3)$
10. $(x^2 - 3x - 2)(2x^2 - x + 4)$

7.7 Closing Ideas

In this section, we saw the last of the algebraic properties of arithmetic.

Definition 7.1. Let a, b, and c be real numbers. Then the following properties hold:

- *The Commutative Property of Multiplication*: $a \cdot b = b \cdot a$
- *The Associative Property of Multiplication*: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

If this looks familiar, it's because we saw something very similar to it a few sections ago. It turns out that addition also has these two properties (Definition 3.5). And when we combine them together with the distributive property (Definition 3.4), we get a pretty thorough description of how arithmetic and algebra works.

It's important to notice that we've never proven any of these properties. We've relied on our experiences going all the way back to when we were first learning basic arithmetic as our foundation. And from those experiences, we've built out our understanding that this is how numbers work. The world would be a very different place if these things weren't true.

- (The commutative property of addition) If we start with an empty bag, then put one apple into bag followed by two apples, we get the same result as if we had put in two apples followed by one apple.
- (The associative property of addition) If we put one apple and two oranges in a bag, and then put three bananas in afterward, we get the same result as if we had put the two oranges and three bananas in the bag first, and then put the one apple in.
- (The commutative property of multiplication) Three bag of five apples have the same number as five bags of three apples.
- (The associative property of multiplication) Two boxes that each contain three bags with four apples has the same number of apples as (two times three) bags with four apples each.

The associative property of multiplication is tricky because we rarely think of what it means to multiply three numbers together.

So at a very basic level, we believe these statements are true because our experiences with reality tell us they should be true. These are not rules that mathematicians came up with and told everyone else they had to follow. Mathematicians simply created a language to describe these things.

When it comes to college level mathematics, it can be helpful to think about the work you're doing with that framework in mind. As much as you can, try to ground all the work that you do in terms of practical reality. You won't always be able to do it. Sometimes, the work that you're doing is an abstraction or generalization of a concept. If that's the case, try to bring yourself back to the last example that made sense, and see if you can use that knowledge to help you build the next piece.

Replacing numbers with variables is an example of an abstraction. Symbolically representing multiplication with a grid is another.

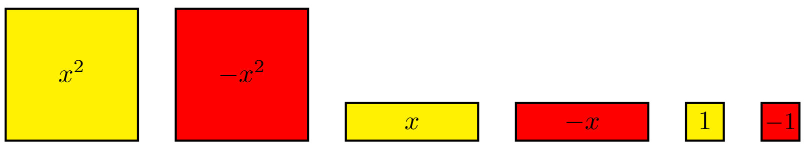
This isn't always going to be easy. Some ideas will take more time to sink in than others. That's part of the nature of learning. But if you invest the time and the energy to slow down and learn to think clearly about what you're doing, you'll have a much better chance at being successful.

7.8 Going Deeper: Algebra Tiles

For many students, one of their struggles with mathematics is that all the concepts feel extremely abstract to them. This can make math particularly difficult for students that are more visual or kinesthetic in their learning. A number of educational manipulatives have been created to help students connect to mathematical thinking, but not all teachers are familiar with them or have encouraged their students to embrace them. We will be discussing a few of these as we work our way through the content to help bring deeper insights into the material.

You can print up paper manipulatives off the internet, but they can sometimes be a bit flimsy on their own. If you do that, you may want to glue them to pieces of cardboard (similar in thickness to a cereal box) to give them some extra thickness. Alternatively, there are collections virtual manipulatives online or you can purchase more durable plastic tiles from online sources.

Algebra tiles were introduced in this section to help visualize products, but they can be used more generally to help represent a range of algebraic concepts. We will start by describing the tiles themselves.

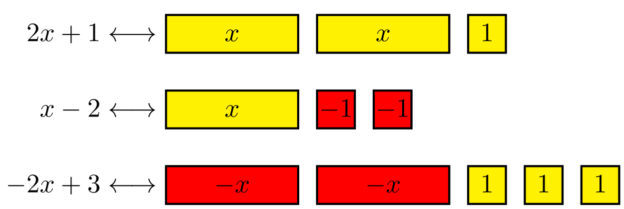


There are three basic shapes of tiles, and each one comes in two colors. The three shapes are a small square, a rectangle, and a large square. The dimensions of the different shapes correspond to each other, so that the narrow side of the rectangle matches with the small square and the long side of the rectangle matches with the large square. The actual relationship between the sides of the small square and large square are irrelevant. They just need visibly different in size. The small square is a unit tile, the rectangle is an x tile, and the large square is an x^2 tile. The two colors represent either a positive or negative version of the various tiles.

Some manufacturers have multiple colors and allow the teachers to define the different meanings for each of them. We're going to stick with just two colors for the bulk of our discussion.

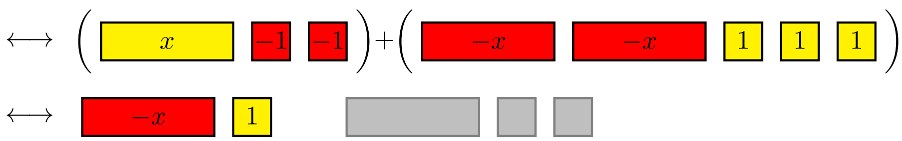
Representing and Simplifying Algebraic Expressions

Algebraic expressions can be represented by collections of tiles. Here are some examples:



Adding algebraic expressions together can be accomplished by combining two collections of tiles together. When combining tiles, there is an additional rule where a positive and negative version of the same shape will cancel each other out. When working with manipulatives, this is accomplished by simply setting the canceled tiles off to the side. For visualization purposes in this book, we will use a light gray outline to represent a pair of canceled tiles.

$(x-2)+(-2x+3)$

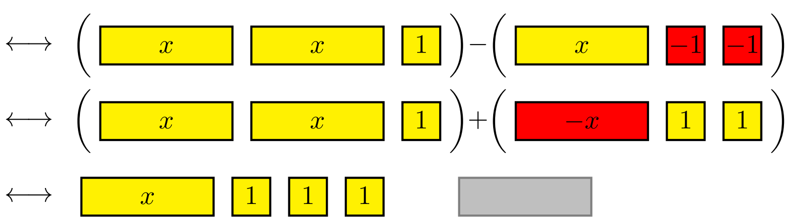


$\longleftrightarrow\ -x+1$

There was one pair of x tiles and two pairs of unit tiles that canceled out. This leaves one negative x tile and one positive unit tile.

Subtracting is similar to addition, except that we first need to swap the colors of the tile. This corresponds to distributing the negative sign across the parentheses.

$(2x+1)-(x-2)$

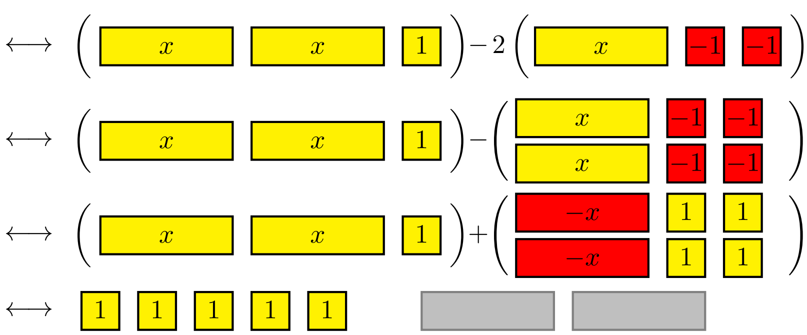


$\longleftrightarrow\ x+3$

This is a tangible representation of "subtraction is addition of the opposite."

The distributive property can be thought of as having multiple bundles of the tiles in the parentheses. If the distributive property is paired with subtraction, we create multiple copies first, then subtract by swapping the colors.

$(2x+1)-2(x-2)$

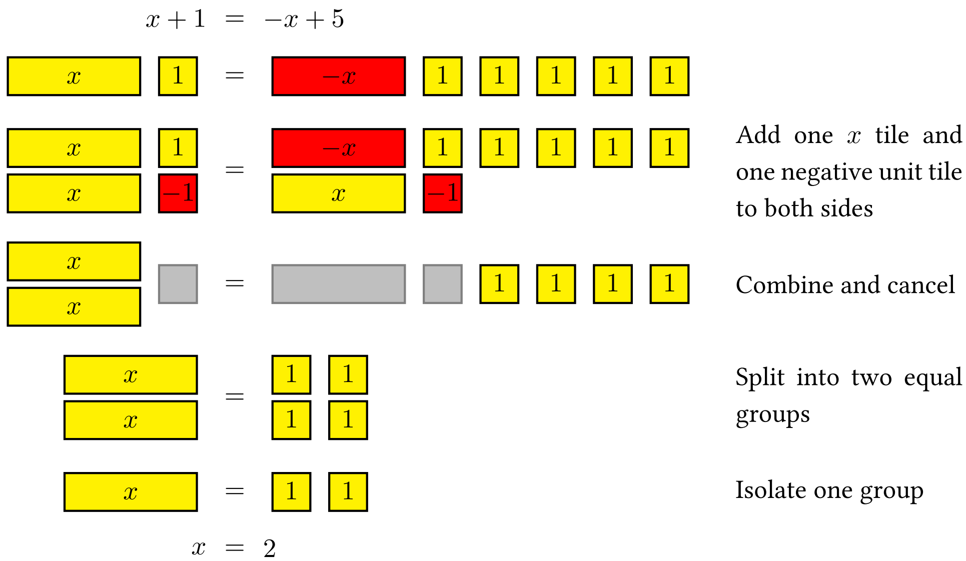


$\longleftrightarrow\ 5$

Solving Equations

The process of solving equations with algebra tiles focuses on the idea that the two quantities on either side of the equation must be the same. On each side, we can use any of the simplification steps from before. In addition to this, there are two other operations. We can add the same set of tiles to both sides, and we can divide the blocks on each side into an equal number of groups.

Here is an example of solving the equation $x+1=-x+5$:



We have also completely removed the canceled pairs of tiles.

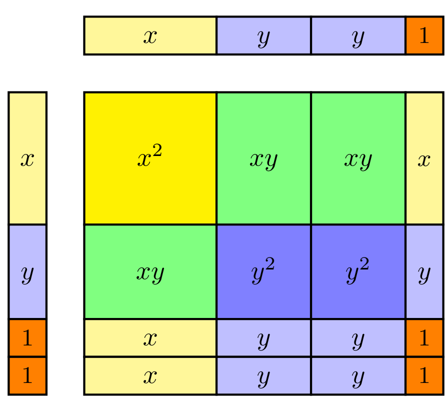
This process directly mirrors the algebraic steps involved in solving equations. You should be able to recognize that adding tiles is either addition or subtraction of algebraic expressions (depending on whether positive or negative tiles are added) and the grouping step is related to division.

We'll take a closer look at the connection between division and making groups of objects in a later section.

Other Uses of Algebra Tiles

These are not the only ideas that algebra tiles can represent. In this section, we briefly mentioned how algebra tiles can be used for representing multiplication. The idea here is to think of the tiles representing the length and width of a rectangle, and then additional tiles can be used to represent the result as an area. We'll later see how tiles can be used to help factor polynomials.

The idea can be taken further by creating an additional length y and making tiles that correspond to products involving that other variable. The diagram below shows how we could represent the product $(x + 2y + 3)(x + y + 2)$ this extended tile set. Note that we changed the color scheme slightly to highlight the different shapes.



$$(x + 2y + 1)(x + y + 2) = x^2 + 3xy + y^2 + 3x + 5y + 2$$

This diagram doesn't even take into account the ways that we can use negative versions of these tiles to write even more polynomial products. However, at a certain point you start to reach diminishing returns on the algebra tiles, and the actual algebra with symbols becomes a better option.

7.9 Solutions to the "Try It" Examples

1

$$\begin{aligned} -3a^3b^2 \cdot 6ab^3 &= (-3 \cdot a \cdot a \cdot a \cdot b \cdot b) \cdot (6 \cdot a \cdot b \cdot b \cdot b) && \text{Definition of exponents} \\ &= -3 \cdot 6 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot b && \text{Rearrange the factors} \\ &= -18a^4b^5 && \text{Definition of exponents} \end{aligned}$$

$$\begin{aligned} -3a^3b^2 \cdot 6ab^3 &= -3 \cdot 6 \cdot a^{3+1} \cdot b^{2+3} && \text{Properties of exponents} \\ &= -18a^4b^5 && \text{Arithmetic} \end{aligned}$$

$$-3a^3b^2 \cdot 6ab^3 = -18a^4b^5 \qquad \text{Multiplying monomials}$$

2

	6
$5x$	$30x$

	$-5xy^2$
$3x^3y$	$-15x^4y^3$

	$-3n^3$
$-7m^2n$	$21m^2n^4$

3

	$2x$	-3
x	$2x^2$	$-3x$
4	$8x$	-12

$$\begin{aligned} (2x-3)(x+4) &= 2x^2 - 3x + 8x - 12 && \text{Distributive property} \\ &= 2x^2 + 5x - 12 && \text{Combining like terms} \end{aligned}$$

4

	x^2	$3x$	-4
x^2	x^4	$3x^3$	$-4x^2$
$-5x$	$-5x^3$	$-15x^2$	$20x$
2	$2x^2$	$6x$	-8

$$\begin{aligned} &(x^2+3x-4)(x^2-5x+2) \\ &\quad = x^4 + 3x^3 - 4x^2 - 5x^3 - 15x^2 + 20x + 2x^2 + 6x - 8 && \text{Distributive property} \\ &\quad = x^4 + 3x^3 - 5x^3 - 4x^2 - 15x^2 + 2x^2 + 20x + 6x - 8 && \text{Rearrange the terms} \\ &\quad = x^4 - 2x^3 - 17x^2 + 26x - 8 \end{aligned}$$

You All Have This in Common: Common Factors

Learning Objectives:

- ☐ Determine the greatest common factor in a polynomial and factor it out.
- ☐ Factor by grouping.

In the previous section, we learned how to multiply polynomials. In this section, we are going to start to learn how to undo everything we just did.

We will begin the process by looking at a basic grid product from the previous section. We started with values around the edges of the grid and used that information to tell us the values inside of the grid.

	x	6
4		

$\longrightarrow$

	x	6
4	$4x$	24

For this section, we are going to start with the numbers in the grid and try to figure out what values go around the edges.

$4x$	24

$\longrightarrow$

	?	?
?	$4x$	24

As it turns out, there are several ways to do this. Here are four examples:

	$4x$	24
1	$4x$	24

	$2x$	12
2	$4x$	24

	x	6
4	$4x$	24

	$\frac{2x}{3}$	4
6	$4x$	24

There is nothing mathematically wrong with any of these products. But we can see that some choices are better than others. The choice on the far left doesn't accomplish anything at all. The one on the far right introduces fractions. So we really want to focus on the middle two.

There's also nothing wrong with fractions. But there are certainly times when we prefer not to work with them.

Of the middle two options, the one on the right can be understood as being the "better" option. The reason for that is that we factored out the biggest amount possible. And that is the basic goal. Technically, we call this the greatest common factor.

Definition 8.1. A *common factor* of the terms of a polynomial is a monomial that divides all of the terms. The *greatest common factor* is the monomial that has the largest degree and largest coefficient.

The degree of a monomial is sum of the exponents of the variable part. So the degree of $3x^2y^4$ is 6. But don't worry too much about this. You should be able to tell from experience after working through the examples.

1 The end result of this manipulation is going to be an expression that is equivalent to the one that we started with. So for the sample grid we've been using, we've been trying to identify the greatest common factor of the polynomial $4x + 24$. Here's what the presentation (including the grid) would look like.

	x	6
4	$4x$	24

$$4x + 24 = 4(x + 6)$$

The grid will become scratch work. For now, it's just part of the practice.

Try it: Identify the greatest common factor of $5x^2 + 15$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

2 Conceptually, there is nothing different about this process when there are more variables and more terms involved. The main challenge is to avoid simple arithmetic errors by miscounting the variables in the expressions when we factor them out.

	$2x$	5
$3x$	$6x^2$	$15x$

$$6x^2 + 15x = 3x(2x + 5)$$

	$3x$	$2y$	-5
$4xy$	$12x^2y$	$8xy^2$	$-20xy$

$$12x^2y + 8xy^2 - 20xy = 4xy(3x + 2y - 5)$$

Try it: Identify the greatest common factor of $6x^2y - 8xy + 14y$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

A slightly more complex version of this concept is known as factoring by grouping. The idea here is that we're going to factor out the greatest common factor of two pairs of terms, then we're going to look at the new terms you've created and see if there's something to factor out there. But this requires us to expand our concept of what a common factor can be.

To visualize this, we will look at a specific example through a substitution.

	x	3
y	xy	$3y$

$$\xrightarrow{y = x + 2}$$

	x	3
$(x + 2)$	$x(x + 2)$	$3(x + 2)$

$$xy + 3y = (x + 3)y$$

$$x(x + 2) + 3(x + 2) = (x + 3)(x + 2)$$

Notice how important the parentheses are for the substitution in the grid.

The key to understanding this is that we can treat terms inside of parentheses as if they were a single object when we're factoring. As long as the objects in the parentheses are identical, we can factor them out like any other variable.

3 Let's take a look at a full example of factoring by grouping. Consider the expression $x^3 + 3x^2 + 2x + 6$. We are going to think of this as two pairs of expressions:

$$x^3 + 3x^2 + 2x + 6 \longrightarrow$$

x^3	$3x^2$

$2x$	6

Each of these grids gives rise to a factorization.

	x	3
x^2	x^3	$3x^2$

	x	3
2	$2x$	6

$$x^3 + 3x^2 = x^2(x+3) \qquad 2x + 6 = 2(x+3)$$

We can then take these results and put that into another grid and look for another factorization.

	x^2	2
$x+3$	$x^2(x+3)$	$2(x+3)$

$$x^2(x+3) + 2(x+3) = (x^2+2)(x+3)$$

The presentation that we've given above is much more focused on understanding the process. The presentation was broken into pieces that facilitate explanation, and a final presentation can be much more compact.

	x	3
x^2	x^3	$3x^2$

	x	3
2	$2x$	6

	x^2	2
$x+3$	$x^2(x+3)$	$2(x+3)$

Again, these grids will become scratch work. In the end, the equations will be what matters.

$$\begin{aligned} x^3 + 3x^2 + 2x + 6 &= x^2(x+3) + 2(x+3) && \text{Factor by grouping} \\ &= (x^2+2)(x+3) && \text{Factor out the common factor} \end{aligned}$$

Try it: Factor $x^3 - 2x^2 + 4x - 8$ by grouping. Draw the grids and use a complete presentation.

4 It is very important to be careful with negative signs, especially when the negative sign is the third term of the polynomial. Students will sometimes make unfortunate groupings that are incorrect.

$$x^2 + 4x - 2x - 8 \stackrel{\times}{=} (x^2 + 4x) - (2x - 8)$$

Can you identify the error?

When thinking about the factorization, it is important to keep the signs with the corresponding terms when you visualize the grid.

	x	4
x	x^2	$4x$

	x	4
-2	$-2x$	-8

	x	-2
$x+4$	$x(x+4)$	$-2(x+4)$

$$\begin{aligned} x^2+4x-2x-8 &= x(x+4)-2(x+4) && \text{Factor by grouping} \\ &= (x-2)(x+4) && \text{Factor out the common factor} \end{aligned}$$

Try it: Factor $x^2-3x-4x+12$ by grouping. Draw the grids and use a complete presentation.

Did you notice that the x terms are not combined? We'll see more of this in the next section.

Not every expression of this type can be factored by grouping. You will need to trust your algebra and recognize when the factoring step does and does not work.

8.1 Common Factors - Worksheet 1

1 Identify the greatest common factor of $6x + 9$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

2 Identify the greatest common factor of $4y - 10$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

3 Identify the greatest common factor of $3a^2b + 9ab - 15b$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

4 Identify the greatest common factor of $8n + 4$, then factor it out of the polynomial. Write the corresponding equation, but do not draw a grid.

In the long run, you will want to be able to factor out terms mentally and go straight to the final result.

8.2 Common Factors - Worksheet 2

1 Identify the greatest common factor of $4x^5 - 10x^2 + 12x$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

2 Identify the greatest common factor of $12p^2 - 18p$, then factor it out of the polynomial. Write the corresponding equation, but do not draw a grid.

3 Identify the greatest common factor of $3y^3 + 15y^2 - 6y$, then factor it out of the polynomial. Write the corresponding equation, but do not draw a grid.

4 Identify the greatest common factor of $x(x-4)+3(x-4)$, then factor it out of the polynomial. Draw the grid and write the corresponding equation.

Remember that the $(x - 4)$ can be viewed as a single object for factoring.

8.3 Common Factors - Worksheet 3

1 Identify the greatest common factor of $x(3y-2)-5(3y-2)$, then factor it out of the polynomial. Write the corresponding equation, but do not draw a grid.

2 Factor $x^3+4x^2+3x+12$ by grouping. Draw the grids and use a complete presentation.

3 Factor $x^2-6x-3x+18$ by grouping. Draw the grids and use a complete presentation.

8.4 Common Factors - Worksheet 4

1 Factor $2x^2 - 3x - 6x + 9$ by grouping. Draw the grids and use a complete presentation.

2 Factor $3xy + 6x - 2y - 4$ by grouping. Draw the grids and use a complete presentation.

3 Factor $x^3 - 3x^2 + 4x - 12$ by grouping. Use a complete presentation, but do not draw the grids.

This is the end goal for your presentations using factoring by grouping.

8.5 Common Factors - Worksheet 5

1 Factor $x^2 - 4x - 4x + 16$ by grouping. Use a complete presentation, but do not draw the grids.

2 Factor $2x^2 + 5x - 4x - 10$ by grouping. Use a complete presentation, but do not draw the grids.

3 Factor $x^2 - 3x + 5x - 15$ by grouping. Then factor $x^2 + 5x - 3x - 15$ by grouping. Use a complete presentation for both, but do not draw the grids. Was one easier than the other? Explain your perspective.

Both factorizations are the same! This is yet another example of there being multiple pathways to the same conclusion in mathematics.

Some students find one grouping easier than the other. You can decide for yourself whether this is true for you.

8.6 Deliberate Practice: Factor by Grouping

Focus on these skills:

- Write the original expression.
- Visualize the grids, but try to do the calculations without drawing them.
- Present your work legibly.

Instructions: Factor by grouping.

1. $x^2 + 3x - 2x - 6$
2. $x^2 - 5x + 3x - 15$
3. $x^2 - 2x - 4x + 8$
4. $x^2 + 4x + 3x + 12$
5. $2x^2 - 8x - 3x + 12$
6. $2x^2 + 5x - 4x - 10$
7. $3x^2 - 6x - 7x + 14$
8. $x^3 + 3x^2 - 2x - 6$
9. $2x^3 - 3x^2 - 6x + 9$
10. $x^3 - 4x^2 + 4x - 16$

8.7 Closing Ideas

Factoring is a basic but important algebraic manipulation. It's nothing more than the distributive property (Definition 3.4) applied backwards.

$$a(b+c) = ab + ac \qquad \text{The Distributive Property}$$
$$ab + ac = a(b+c) \qquad \text{Factor out the common factor}$$

And if there were some aspects of this section that feels familiar, it's because this is the exact same algebra that is used with combining like terms.

Being able to relate old ideas to new ideas is an important aspect of mathematical thinking.

$$3x + 4x = (3+4)x = 7x \qquad \text{Combine like terms}$$
$$x^2 + 4x = (x+4)x \qquad \text{Factor out the common factor}$$

The difference between combining like terms and factoring out the common factor is that there's an extra step of arithmetic that we can do with numbers that we can't do with algebraic expressions.

Math is an extremely scaffolded subject. What this means is that new ideas are very often built on older ones. This also means that weaknesses in the foundation make the higher levels of mathematical thinking less stable. Hopefully, as you've been working your way through these sections, you have been taking the time to think through the ideas and solidify those core concepts.

8.8 Solutions to the "Try It" Examples

1

	x^2	3
5	$5x^2$	15

$$5x^2 + 15 = 5(x^2 + 3)$$

2

	$3x^2$	$-4x$	7
$2y$	$6x^2y$	$-8xy$	$14y$

$$6x^2y - 8xy + 14y = 2y(3x^2 - 4x + 7)$$

3

	x	-2
x^2	x^3	$-2x^2$

	x	-2
4	$4x$	-8

	x^2	4
$x-2$	$x^2(x-2)$	$4(x-2)$

$$\begin{aligned} x^3 - 2x^2 + 4x - 8 &= x^2(x-2) + 4(x-2) && \text{Factor by grouping} \\ &= (x^2+4)(x-2) && \text{Factor out the common factor} \end{aligned}$$

4

	x	-3
x	x^2	$-3x$

	x	-3
-4	$-4x$	12

	x	-4
$x-3$	$x(x-3)$	$-4(x-3)$

$$\begin{aligned} x^2 - 3x - 4x + 12 &= x(x-3) - 4(x-3) && \text{Factor by grouping} \\ &= (x-4)(x-3) && \text{Factor out the common factor} \end{aligned}$$

Section 9

Lots of Methods, Not All Good: Factoring Quadratic Polynomials

Learning Objectives:

- ☐ Represent factoring quadratic polynomials geometrically.
- ☐ Factor monic quadratic polynomials by grouping.
- ☐ Factor non-monic quadratic polynomials by grouping.

Factoring quadratic polynomials is generally seen as an important marker for a student's algebra skill level. Because of this, various teachers have tried to design particular methods to help students factor correctly. Unfortunately, many of those methods are answer-oriented, which leads to students getting into the habit of making incorrect algebraic manipulations.

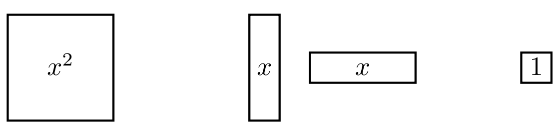
In this section, the focus is going to be on understanding the process of factoring and the ideas behind it. Although getting the right answer is also important, the right answer is the outcome of a proper understanding of the concepts, not the ultimate end in itself.

To emphasize this point, we're going to use a purely geometric framework and sidestep all of the algebra. A couple sections ago, we introduced algebra tiles. These are geometric images that represent different quantities to help us think about algebraic relationships.

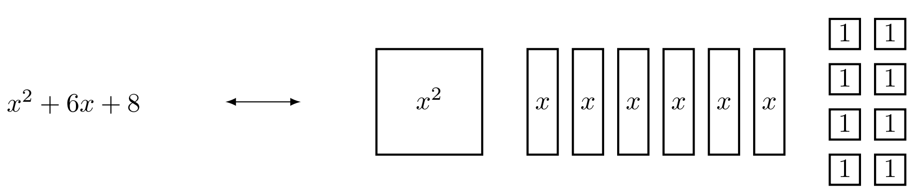
If you want physical manipulatives, you can purchase algebra tiles or make them out of paper.

The basic idea is that you have positive and negative versions of the tiles, and one positive and one negative of the same type of tile will cancel each other out.

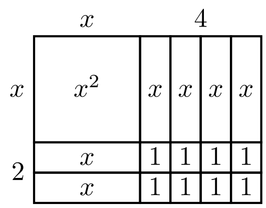
Algebra tiles come with a set of "rules" for how they can be manipulated, which is especially important when you're working with negative values. However, for the purposes of illustration, we're just going to focus on the situation where all of the values are positive. This means that we're going to be thinking about three different tiles (both x tiles should be viewed as the same piece, just rotated relative to each other):



A representation of a quadratic polynomial using the algebra tiles is simply a matter of having the right number of pieces:



The challenge is then to arrange all of these pieces into a rectangle where the small squares are at the bottom right, the large square (or squares) are on the top left, and the rectangles fill in the spaces on the top right and lower left. Once a rectangle is found, it can be used to determine the factorization.



$$x^2 + 6x + 8 = (x + 4)(x + 2)$$

You should count up the pieces and double check that this works.

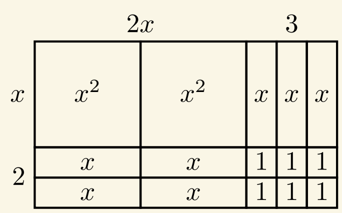
1 We're not going to focus on trying to give you a "method" for finding these rectangles. We're going to trust that your geometric intuition will lead you to the appropriate conclusion.

Try it: Use a diagram of algebra tiles to factor $x^2 + 7x + 12$. Draw the diagram and write the final equation.

The small squares must form a rectangle by themselves, otherwise the tiles won't fit together properly.

Definition 9.1. A quadratic polynomial of the form $ax^2 + bx + c$ is called a *monic* quadratic polynomial if $a = 1$. If $a \neq 1$, then we call it a *non-monic* quadratic polynomial.

2 Algebra tiles can be used to factor both monic and non-monic polynomials. It all comes down to finding the right rectangle.



$$2x^2 + 7x + 6 = (2x + 3)(x + 2)$$

Try it: Use a diagram of algebra tiles to factor $2x^2 + 9x + 4$. Draw the diagram and write the final equation.

While the use of diagrams (or physical manipulatives) may help us to get answers, it doesn't necessarily help us to think in an organized and logical manner. So we are going to look at a structured algebraic approach to the same challenge.

But before we do that, let's take a look at why factoring is difficult. Let's take a look at the steps of expanding out a product.

$$(2x + 3)(x + 2) = 2x^2 + 4x + 3x + 6 \qquad \text{Distributive property}$$

We're not going to draw out the grid for these products. But if you want to, you can.

At this point, you can still factor by grouping to get back the original product.

$$\begin{aligned} 2x^2 + 4x + 3x + 6 &= 2x(x + 2) + 3(x + 2) && \text{Factor by grouping} \\ &= (2x + 3)(x + 2) && \text{Factor out the common factor} \end{aligned}$$

The challenge arises once the like terms are combined. We no longer have access to the rationale behind un-combining the like terms. In fact, there are lots of ways to un-combine like terms!

$$\begin{aligned} 2x^2 + 7x + 6 &= 2x^2 + x + 6x + 6 \\ &= 2x^2 + 2x + 5x + 6 \\ &= 2x^2 + 3x + 4x + 6 \end{aligned}$$

These examples don't even take into account possibilities with negative numbers!

Some students learn how to do this with monic polynomials with a guessing method, but it very often falls apart on them when they are working on non-monic polynomials. At the core, most of the problems arise from the students learning to simply "write down" their answer without understanding how or why it worked. We want to avoid that type of method because it does not enhance mathematical thinking.

There are some methods that lead students to write down expressions that are mathematical nonsense, and then do some manipulations from there to get the answer.

We will be using the "ac method" of factoring. The goal is to uncover the right way to un-combine like terms so that we can factor by grouping. Here is a representation of the method:

Here are a few of the reasons for this choice:

1. All the steps are algebraically valid.
2. The same technique is used for both monic and non-monic quadratics.
3. It builds on the same intuition as most of the other methods.

$$ax^2 + bx + c \longrightarrow \begin{cases} \text{Multiply to } ac \\ \text{Add to } b \end{cases}$$

As the text states, the goal is to find two numbers that multiply to ac and add to b. The values you obtain from this will be the way to un-combine the like terms, which will allow you to factor by grouping.

3 We will start by applying this method to factor $x^2 - 3x - 10$. We will first translate our quadratic into the two target properties.

$$x^2 - 3x - 10 \longrightarrow \begin{cases} \text{Multiply to } -10 \\ \text{Add to } -3 \end{cases}$$

Notice that $a = 1$, $b = -3$, and $c = -10$. This means that $ac = 1 \cdot (-10) = -10$.

And here is where you simply have to use your experience with numbers to try to find the right combination. What two numbers multiply to -10 and add to -3? You should be able to determine that the values are -2 and 5. These two values are used to un-combine the like terms ($-3x = -5x + 2x$), which then allows us to complete the factorization using grouping. Here is the complete presentation.

Hint: It's usually easier to focus on the multiplication pairs because there are fewer of them.

$$\begin{aligned} x^2 - 3x - 10 &= x^2 - 5x + 2x - 10 && \text{The } ac \text{ method} \\ &= x(x-5) + 2(x-5) && \text{Factor by grouping} \\ &= (x+2)(x-5) && \text{Factor out the common factor} \end{aligned}$$

Try it: Use the ac method to factor $x^2 - 3x - 10$ using a complete presentation, but instead of un-combining the middle term using $-3x = -5x + 2x$, swap the order and un-combine it using $-3x = 2x - 5x$.

Whether you write out the "multiply to" and "add to" statements is up to you. Some students find it helpful for organization, while others find it tedious and distracting. In any case, that should be considered scratch work.

You should get the same final answer no matter which way you do it.

One of the main values of the ac method is that it works for both monic and non-monic polynomials. This benefit allows us to simplify the process of factoring quadratic polynomials to a single approach that works for all situations. So by adopting the ac method, we don't have to create two different sets of methods to accomplish the same goal.

4 We will use this method to factor $4x^2 + 5x - 6$.

$$4x^2 + 5x - 6 \longrightarrow \begin{cases} \text{Multiply to } -24 \\ \text{Add to } 5 \end{cases}$$

In this case, $a = 4$, $b = 5$, and $c = -6$, so $ac = 4 \cdot (-6) = -24$.

With a little bit of thought, you will find the combination of 8 and -3 will satisfy the goal.

$$\begin{aligned} 4x^2 + 5x - 6 &= 4x^2 + 8x - 3x - 6 && \text{The } ac \text{ method} \\ &= 4x(x+2) - 3(x+2) && \text{Factor by grouping} \\ &= (4x-3)(x+2) && \text{Factor out the common factor} \end{aligned}$$

Try it: Use the ac method to factor $2x^2 + 5x - 12$ using a complete presentation.

9.1 Factoring Quadratic Polynomials - Worksheet 1

1 Translate the diagram of algebra tiles into an equation.

x^2	x	x	x	x	x
x	1	1	1	1	1
x	1	1	1	1	1
x	1	1	1	1	1
x	1	1	1	1	1

2 Use a diagram of algebra tiles to factor $x^2 + 8x + 15$. Draw the diagram and write the final equation.

You may want to do some actual scratch work on scratch paper for this.

3 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$x^2 + 8x + 15 \longrightarrow$ { Multiply to ☐ ; Add to ☐ }

9.2 Factoring Quadratic Polynomials - Worksheet 2

1 Use a diagram of algebra tiles to factor $2x^2 + 9x + 4$. Draw the diagram and write the final equation.

Put both of your x^2 tiles on the top left corner.

2 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$$x^2 + 2x - 8 \longrightarrow \begin{cases} \text{Multiply to } \boxed{} \\ \text{Add to } \boxed{} \end{cases}$$

3 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$$2x^2 - 3x - 28 \longrightarrow \begin{cases} \text{Multiply to } \boxed{} \\ \text{Add to } \boxed{} \end{cases}$$

9.3 Factoring Quadratic Polynomials - Worksheet 3

1 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$$x^2 - 8x + 7 \longrightarrow \begin{cases} \text{Multiply to } \boxed{} \\ \text{Add to } \boxed{} \end{cases}$$

2 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$$2x^2 + 5x + 2 \longrightarrow \begin{cases} \text{Multiply to } \boxed{} \\ \text{Add to } \boxed{} \end{cases}$$

3 Use the ac method to factor $x^2 - 7x + 10$ using a complete presentation.

You can continue to write out the two conditions if you want. Eventually, that will be a mental exercise

9.4 Factoring Quadratic Polynomials - Worksheet 4

1 Fill in the appropriate value into the boxes, then use the ac method to factor the given quadratic polynomial using a complete presentation.

$$3x^2 - 10x - 8 \longrightarrow \begin{cases} \text{Multiply to } \boxed{} \\ \text{Add to } \boxed{} \end{cases}$$

2 Use the ac method to factor $x^2 + 6x + 9$ using a complete presentation.

3 Use the ac method to factor $x^2 - 3x - 40$ using a complete presentation.

4 Use the ac method to factor $2x^2 - 5x - 3$ using a complete presentation.

9.5 Factoring Quadratic Polynomials - Worksheet 5

1 Use the ac method to factor $x^2 + 9x + 20$ using a complete presentation.

2 Use the ac method to factor $4x^2 - 4x - 3$ using a complete presentation.

3 Factor $x^2 + 10x + 16$ and $x^2 - 10x + 16$, then compare the results. What do you notice about the factorizations?

Looking for patterns is a core element of mathematical thinking.

4 Factor $x^2 - 5x - 14$ and $x^2 + 5x - 14$, then compare the results. What do you notice about the factorizations?

9.6 Factoring Quadratic Polynomials - Worksheet 6

1 Suppose you are trying to factor a quadratic that has the following condition:

$$\begin{cases} \text{Multiply to a positive number} \\ \text{Add to a positive number} \end{cases}$$

What information can you conclude about the signs of the two numbers you're looking for?

If you're not sure, look back at some of the previous problems that meet this criteria and see if you can find the common feature.

2 Suppose you are trying to factor a quadratic that has the following condition:

$$\begin{cases} \text{Multiply to a positive number} \\ \text{Add to a negative number} \end{cases}$$

What information can you conclude about the signs of the two numbers you're looking for?

3 Suppose you are trying to factor a quadratic that has the following condition:

$$\begin{cases} \text{Multiply to a negative number} \\ \text{Add to a positive number} \end{cases}$$

What information can you conclude about the signs of the two numbers you're looking for?

4 Suppose you are trying to factor a quadratic that has the following condition:

$$\begin{cases} \text{Multiply to a negative number} \\ \text{Add to a negative number} \end{cases}$$

What information can you conclude about the signs of the two numbers you're looking for?

9.7 Deliberate Practice: Factoring Quadratic Polynomials

Focus on these skills:

- Write the original expression.
- Visualize the grids, but try to do the calculations without drawing them.
- Mentally state the “Multiply to” and “Add to” properties. You can write them out as scratch work, if necessary.
- Present your work legibly.

Instructions: Use the ac method to factor using a complete presentation.

1. $x^2 + 2x - 15$
2. $x^2 - 4x + 4$
3. $x^2 + 7x + 12$
4. $x^2 - 9$
5. $x^2 + 8x + 16$
6. $x^2 - x - 42$
7. $2x^2 + 5x - 3$
8. $3x^2 - 13x - 10$
9. $2x^2 - 3x - 20$
10. $4x^2 + 4x - 15$

9.8 Closing Ideas

As mentioned earlier, factoring quadratic polynomials is viewed as an important marker of your algebraic skill level. But the skill isn't just in getting the answer. In fact, the bulk of mathematics is not about getting the answer. If you look back over everything we've done so far, you will see that there is a heavy emphasis on understanding and communicating the thought processes involved in performing these algebraic manipulations.

On the last worksheet page, instead of having you do more factorizations, you were asked to seek out patterns in the factorizations you've already done. You were only asked to identify the information you could conclude in each situation, but you were hopefully also able to discover the underlying logic. That logic goes all the way back to basic arithmetic, with ideas such as the following:

- A positive number multiplied by a positive number is a positive number.
- A negative number multiplied by a negative number is a positive number.
- A positive number multiplied by a negative number is a negative number.
- A negative number multiplied by a positive number is a negative number.

Do you know why these are true? Or is it just another thing that you memorized at some point?

There are also ideas that perhaps we don't have "memorized" phrases for, but should make sense when you think about it.

- A positive number plus a positive number is a positive number.
- A negative number plus a negative number is a negative number.
- When adding a positive and a negative number together, the sign of the result will match the sign of the number that is larger (in absolute value)

The third one may feel a bit tricky if you don't slow down. Remember the skill of thinking about specific numbers. Pick a large positive number and add a small negative to it. What is the sign of the result?

Factoring quadratic polynomials is not an end in and of itself. It is a tool that allows you to do advanced algebraic manipulations, such as solving more complicated equations. So we usually do not use it in complete isolation. There is an important property of numbers that can be a useful supplemental tool.

> **Definition 9.2.** The *zero product property* states that if $a \cdot b = 0$, then we must have either $a = 0$ or $b = 0$ (or possibly both).

Think of two non-zero numbers and multiply them together. Is the result ever zero?

The combination of factoring with the zero product property allows you to take an equation with higher degree and convert it into multiple equations of lower degree. Here is an example of solving the equation $x^2 + 3x - 4 = 0$:

$$\begin{array}{cccl} & x^2 + 3x - 4 = 0 & & \\ & (x-1)(x+4) = 0 & & \text{Factor} \\ x - 1 = 0 & \text{or} & x + 4 = 0 & \text{The zero product property} \\ x = 1 & & x = -4 & \text{Solve each equation} \end{array}$$

Notice how the factorization was reduced to just a single step and that none of the steps for solving the equation were justified. When you get further along in your math courses, you will be expected to know how to do many of those steps for yourself, which is why we're strongly emphasizing the steps here. We are creating a foundation for you to build on in the future.

In this section, we can see the full scaffolded nature of algebraic reasoning. The skill of factoring is right in the middle of the tower. Below us, we see that everything we're doing is premised on having basic fluency with algebraic concepts such as the distributive property, and it also requires us to have enough experience with numbers to search through different combinations to find the one we want. Above us, we see that there's an entirely new set of equations that we can solve once we add in some more ideas. Every time you learn a new idea in mathematics, you should take a step back to see how it fits into the bigger picture.

9.9 Going Deeper: Special Factorizations

There are certain factoring patterns that are useful to learn to recognize. It's not that you would be unable to factor these without recognizing them, but they come up so frequently that they have special names so that we can identify them when they happen:

Square of a binomial sum:	$a^2 + 2ab + b^2 = (a+b)^2$
Square of a binomial difference:	$a^2 - 2ab + b^2 = (a-b)^2$
Difference of squares:	$a^2 - b^2 = (a+b)(a-b)$

The basic idea of using these patterns to factor is that you need to match your expression with one of the above formulas and identify the appropriate values of a and b. For example, suppose you want to factor $x^2 + 6x + 9$ and you want to check whether this fits one of the patterns above. By counting the number of terms and looking at the signs, we can see that there's only one formula that has a chance.

Square of a binomial sum	Square of a binomial difference	Difference of Squares
$\begin{array}{ccccc} a^2 & + & 2ab & + & b^2 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ x^2 & + & 6x & + & 9 \end{array}$	$\begin{array}{ccccc} a^2 & - & 2ab & + & b^2 \\ \updownarrow & \times & \updownarrow & \updownarrow & \updownarrow \\ x^2 & + & 6x & + & 9 \end{array}$	$\begin{array}{ccccc} a^2 & & - & & b^2 \\ \updownarrow & \times & \times & \times & \updownarrow \\ x^2 & + & 6x & + & 9 \end{array}$

Once we know that we should be focusing on the square of a binomial sum, we can try to identify the a and the b by looking at the terms on the end and then check to see if the term in the middle matches. We can see that we can take $a = x$ and $b = 3$, which means that the middle term should be $2ab = 6x$. And this all matches perfectly.

$$x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2 = (x+3)^2$$

The nice part about these formulas is that it gives us insights into other factorizations that we may not immediately see using the ac method. Here's an example:

$$x^2 + 2\sqrt{2}x + 2 = x^2 + 2 \cdot x \cdot \sqrt{2} + (\sqrt{2})^2 = (x + \sqrt{2})^2$$

It's not that the ac method can't be applied here. Here's how it would look.

$$x^2 + 2\sqrt{2}x + 2 \longrightarrow \begin{cases} \text{Multiply to } 2\sqrt{2} \\ \text{Add to } 2 \end{cases}$$

It's technically true that $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ and $\sqrt{2} \cdot \sqrt{2} = 2$, and we would be able to do the factorization by grouping:

$$\begin{aligned} x^2 + 2\sqrt{2}x + 2 &= x^2 + \sqrt{2}x + \sqrt{2}x + 2 && \text{The } ac \text{ method} \\ &= x(x+\sqrt{2}) + \sqrt{2}(x + \sqrt{2}) && \text{Factor by grouping} \\ &= (x+\sqrt{2})(x+\sqrt{2}) && \text{Factor out the common factor} \end{aligned}$$

But we can see from this that the ac method is really designed for thinking through integer factorizations. Recognizing the factorization with square roots is just not something we should expect students at this level to do.

The reason we use these special factorizations is because they are a special pattern that we can learn to recognize with practice. And that's pretty much all this is. By learning to recognize these patterns, you have access to a larger collection of factorizations that you might otherwise miss. Here are some examples:

You should check these factorizations by working out the products.

Square of a binomial sum

$$\begin{aligned} x^2+6x+9 &= (x+3)^2 \\ x^2+2xy+y^2 &= (x+y)^2 \\ x^4+10x^2+25 &= (x^2+5)^2 \end{aligned}$$

Square of a binomial difference

$$\begin{aligned} x^2-12x+36 &= (x-6)^2 \\ x^2-4xy+4y^2 &= (x-2y)^2 \\ x^6-2x^3y^2+y^4 &= (x^3-y^2)^2 \end{aligned}$$

Difference of squares

$$\begin{aligned} x^2-49 &= (x+7)(x-7) \\ 4x^2-9y^2 &= (2x+3y)(2x-3y) \\ x^2-2 &= (x+\sqrt{2})(x-\sqrt{2}) \end{aligned}$$

Another aspect of these formulas is that your awareness of them can help you to avoid a few very specific algebraic errors. If you are familiar with the formulas for the square of a binomial sum and difference, you would immediately recognize what these expressions should be. These are errors that are so common that one of them has been given the nickname of "the freshman's dream."

Freshmen dream that this is true because it would keep them from losing so many points!

$$(a+b)^2 \overset{\times}{=} a^2+b^2 \qquad\qquad (a-b)^2 \overset{\times}{=} a^2-b^2$$

You might have noticed that that there's a formula for the square of a binomial sum and difference, but only a formula for the difference of squares and not a sum of squares. It turns out that the sum of squares cannot be factored using the tools that we have developed so far. We can use logic to prove that we can't do this. Let's try to factor x^2+9 as an example. Using the ac method, we have to find values that do the following:

This logic works for anything of the form x^2+y^2.

$$x^2+9 \longrightarrow \begin{cases} \text{Multiply to } 9 \\ \text{Add to } 0 \end{cases}$$

In order to get two numbers to multiply to a positive number, they must either both be positive or both be negative. But if they are both positive or both negative, then when you add them together, you can't get zero because both numbers have the same sign. And so there aren't any numbers that will do this for us.

At least, none of the real numbers would work. But that hasn't stopped mathematicians from imagining new types of numbers that could do this. (Pun intended.)

The special factorization formulas don't stop there. Factorization is such an important concept in mathematics that there are all sorts of factorizations that can be helpful at various times and various situations. Here are a few examples, all of which you can check by multiplying everything out:

- The sum of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- The difference of cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
- The difference of fourth powers: $a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$
- A "magic" formula: $x^2 + bx + c = \left(x - \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}\right)\right)\left(x + \left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}\right)\right)$

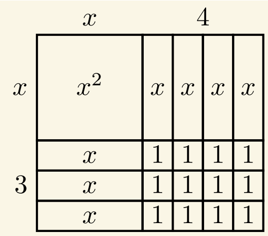
Do you see the pattern in the difference of powers?

The last formula is not one that you're likely to find anywhere, but it may look similar to something you might remember from your past mathematical experiences. And it's actually related to an approach to factoring that was known by mathematicians over 3500 years ago that was introduced to the modern world by Dr. Po-Shen Loh in 2019.

You can find several understandable explanations of this alternative factoring technique on the internet.

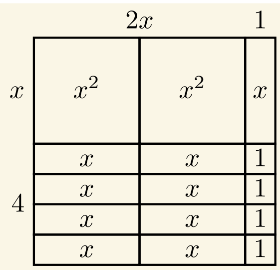
9.10 Solutions to the "Try It" Examples

1



$$x^2 + 7x + 12 = (x + 4)(x + 3)$$

2



$$2x^2 + 9x + 4 = (2x + 1)(x + 4)$$

3

$$\begin{aligned} x^2 - 3x - 10 &= x^2 + 2x - 5x - 10 && \text{The } ac \text{ method} \\ &= x(x + 2) - 5(x + 2) && \text{Factor by grouping} \\ &= (x - 5)(x + 2) && \text{Factor out the common factor} \end{aligned}$$

4

$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 - 3x + 8x - 12 && \text{The } ac \text{ method} \\ &= x(2x - 3) + 4(2x - 3) && \text{Factor by grouping} \\ &= (x + 4)(2x - 3) && \text{Factor out the common factor} \end{aligned}$$

So many symbols! Reading Mathematical Expressions

Learning Objectives:

- ☐ Correctly write and interpret algebraic expressions.
- ☐ Recognize common algebraic errors.
- ☐ Solve linear equations

As we continue forward into higher levels of mathematics, we will encounter more mathematical notation. Students sometimes start to struggle because they've never developed a set of tools to help them interpret those symbols. It's like trying to read a language when you don't have a sense of the grammar. You can sometimes piece things together and follow what's happening, but most of the time you're completely lost.

We've mentioned before that math can be understood as a language. This is just another example of that.

We will begin with familiar territory. At some point in their lives, most students learned some version of the order of operations (often called PEMDAS). This is a set of instructions for the correct way to perform a complex calculation by prioritizing certain calculations. Here is how the order of operations is usually presented:

- Parentheses and other grouping symbols.
- Exponents
- Multiplication and division (performed left-to-right)
- Addition and subtraction (performed left-to-right)

Unfortunately, some students learn this as "multiplication THEN division" and "addition THEN subtraction" which is incorrect.

This is fine as far as the basics are concerned. And when working with numbers, students are usually able to do this correctly. But it's not long after we start introducing variables that students can feel overwhelmed by symbols.

The best analogy for understanding this comes from thinking about a complex sentence in English: "The man carrying a large bag containing three apples and two pears bought a blue shirt with six silver buttons from the store on the corner of Main Street and West Park Avenue." There are a lot of details in that sentence, but there's also the "big picture" of the sentence. What is actually happening in the sentence? Some man bought a shirt from a store. From there we can go into specific details, such as what the man was holding, the color of the shirt, and the location of the store. But those details add information to the big picture without changing the core of the picture.

The focus of this section is not really on algebra, but on helping you familiarize yourself with the language of mathematics. Fortunately, mathematicians have created an entire notation to help you do this. Consider the following polynomial:

$$x^3 - 3x^2 - 7x + 11$$

At this point, your experience should lead you to interpret this as the sum of four terms. You should be able to visualize the plus and minus signs as being separators for the different terms.

Compare this to how we would have to write it if we had to write everything out:

$$x \cdot x \cdot x + (-3) \cdot x \cdot x + (-7) \cdot x + 11$$

We still have to use order of operations with multiplication and addition.

It is much harder for our brains to process this because we have to do a lot of extra work. It's harder to locate the plus signs from among all of the symbols, and we have to count out all of the products to figure out how many factors of x there are.

The two key features here are the use of implied multiplication and exponents. These two bits of notation help us to visually condense the information and make it easier to read. The exponents also mean that we don't have to count out the products. We also have the convention that subtraction means addition of the opposite, which lets us write the negative coefficients as subtraction.

Implied multiplication is when we write two symbols next to each other to indicate multiplication instead of using a multiplication symbol between them.

All of this notation helps us to identify that the "big picture" of this is that we have a sum of four terms. From there, we can choose to look at the details of those terms to see (for example) if there are any like terms that we might want to combine.

Another mathematical notation that was developed is the use of fractions for division. Interestingly, it's very rare in the modern mathematical world to use the ÷ symbol. It is not as rare, but uncommon to use / to represent division as well. These tend to cause more confusion than clarity, which is why modern mathematics prefers the use of a fraction. This way, we can clearly distinguish between the numerator and denominator of a fraction. For example, consider the following expression: $x^2 + 3/x + 2$. The strict (and proper) application of the order of operations is that this result is the same as $x^2 + \frac{3}{x} + 2$. But students will write that to mean $\frac{x^2+3}{x+2}$. If you insist on writing your fractions with a diagonal slash, you will need to use parentheses to separate out the numerator and the denominator: $(x^2 + 3)/(x + 2)$.

If you've ever seen the meme about calculating $8 \div 2(2 + 2)$, then you've seen how the ÷ symbol is problematic. It's not so much a math problem as it is a pedantry problem.

Note: Please do not insist on using the diagonal slash

For this reason, students are strongly urged to not use the diagonal slash for division, but write their division using a long horizontal bar (long enough to span the width of all the terms in the numerator and the denominator) so that it is clear what is in the numerator and what is in the denominator.

Similarly, when using the square root, it is important that the bar of the square root is long enough (and only long enough) to cover the parts inside the square root. For example, $\sqrt{x+3}$ is not the same as $\sqrt{x} + 3$, and $\sqrt{x+}\,3$ doesn't even make sense. So please be careful with what you're writing.

1 A basic skill when learning to read mathematical notation is to mentally "group together" things to be able to process what's happening. For example, the polynomial from before can be seen as the sum of four terms.

$$x^3 - 3x^2 - 7x + 11 \longrightarrow \boxed{x^3} + \boxed{(-3x^2)} + \boxed{(-7x)} + \boxed{11} \longrightarrow \boxed{A} + \boxed{B} + \boxed{C} + \boxed{D}$$

Similarly, we can think of fractions as being basically a numerator divided by a denominator.

$$\frac{x^2+3}{x+2} \longrightarrow \frac{\boxed{x^2+3}}{\boxed{x+2}} \longrightarrow \frac{\boxed{A}}{\boxed{B}}$$

Try it: Consider the expression $\frac{x+3}{x-2} + 3x$. Describe the "big picture" perspective of the expression and put boxes around the terms as appropriate.

There is no standard language for this. But potential examples are "this is a division calculation" or "this is a sum."

After seeing the "big picture" we can then break things down further. Once we're inside of the numerator or the denominator of the fraction, and that may consist of multiple terms.

$$\frac{x^2+3}{x+2} \longrightarrow \frac{\boxed{x^2+3}}{\boxed{x+2}} \longrightarrow \frac{\boxed{\boxed{x^2}+\boxed{3}}}{\boxed{\boxed{x}+\boxed{2}}} \longrightarrow \frac{\boxed{\boxed{A}+\boxed{B}}}{\boxed{\boxed{C}+\boxed{D}}}$$

And we can go deeper into the individual terms and break them up into the coefficient (which is sometimes an unwritten 1) and the variable part. And the variable part can be broken down into individual variables. But at this point, it's like focusing on the number of apples in the bag and you've lost sight of the big picture.

It's rare that you have to diagram a sentence to understand it. Sometimes, you may need to reread a portion of it, and the same is going to be true of reading mathematical sentences.

But so far, we've only focused on the basic arithmetic operations. The next large piece of language you will encounter are functions. We aren't going to go into the details of what functions are here. We're just going to use the notation to help you understand how to think about it when you get to them.

The basic shape of a function is $f(x)$. This represents a number just like the variable y would represent a number. The only difference is that with a function, there is a "rule" that tells you how to calculate the number. When we write $f(x)$, we are saying to use the rule associated with f to do the calculation. In addition to the letters f and g (which are used for generic functions), there are also a number of special functions that have names. Here are a few examples:

Name	sine	cosine	tangent	common logarithm	natural logarithm	exponential
Notation	$\sin()$	$\cos()$	$\tan()$	$\log()$	$\ln()$	$\exp()$

The empty parentheses indicate that there is a value to "plug into" the function. So there would normally be some expression inside of those parentheses, which can be a number (such as with $\ln(2)$), a variable (such as with $\sin(x)$), or a variable expression (such as $\exp(-x^2+1)$).

At this point, it doesn't matter what these functions represent, other than they represent a method for calculating some specific number. So it doesn't matter that $\ln(2) \approx 0.693$, only that $\ln(2)$ is some number that your calculator can calculate for you.

At least in terms of algebra, functions behave very similarly to variables. Here are some examples:

$$2f(x) + 3f(x) = 5f(x)$$
$$x\ln(3) - x = (\ln(3) - 1)x$$

Think of "factoring out" the appropriate symbols.

The important note is that the function name cannot be separated from its argument (the parentheses and everything inside of the parentheses). All of that notation should be seen as a single object. That is, you cannot factor out the "f" in the first example:

$$2f(x) + 3f(x) \stackrel{\times}{=} f(2(x) + 3(x))$$

It may help for you to put a box around the function and its argument.

In practice, you will probably just do this mentally.

$$2f(x) + 3f(x) = 2\boxed{f(x)} + 3\boxed{f(x)}$$

The error of breaking apart the function from its argument is similar to other errors where students "distribute" incorrectly:

In fact, all of these errors are of the same type. They are all misinterpretations of function notation.

$$\begin{aligned}
\frac{1}{a+b} &\overset{\times}{=} \frac{1}{a} + \frac{1}{b} \\
(x+y)^2 &\overset{\times}{=} x^2 + y^2 \\
\sqrt{n+m} &\overset{\times}{=} \sqrt{n} + \sqrt{m} \\
\sqrt{n^2+m^2} &\overset{\times}{=} n + m \\
\sin(\theta+\phi) &\overset{\times}{=} \sin(\theta) + \sin(\phi)
\end{aligned}$$

It all comes down to not understanding what the symbols mean or how they behave.

2 Students often feel intimidated by the notation. One trick is to think of the the function as being hidden inside of a box. This will "hide" the details so that you can focus on the big picture.

This is a lot like ignoring the details what was in the man's bag in the complex sentence we used earlier.

$$\begin{aligned}
x\ln(3) + 4 &= \ln(6) && \\
x \cdot \boxed{A} + 4 &= \boxed{B} && \text{Substitute} \\
x \cdot \boxed{A} &= \boxed{B} - 4 && \text{Subtract 4 from both sides} \\
x &= \frac{\boxed{B} - 4}{\boxed{A}} && \text{Divide both sides by } \boxed{A} \\
x &= \frac{\ln(6) - 4}{\ln(3)} && \text{Substitute}
\end{aligned}$$

Try it: Solve the equation $x\sin(1) + 5 = 3x - \cos(2)$. Do it once using a substitution similar to the example above, and then do it without that substitution. Use a complete presentation both times.

It is more traditional to use variables to replace expressions rather than boxes. However, students tend to connect better to the boxes than variables, at least at first. The long-term goal is to not need to use substitutions at all.

3 When we solve equations, sometimes we're not solving for the variable, but some expression involving the variable. This is often the case when the variable is wrapped up in a function. The same type of process can be done by making a substitution similar to the ones above. It is

important to remember to substitute back to the original variable if you do this.

$$
\begin{aligned}
2\sin(x) + \sqrt{2} &= 0 \\
2y + \sqrt{2} &= 0 && \text{Substitute } y = \sin(x) \\
2y &= -\sqrt{2} && \text{Subtract } \sqrt{2} \text{ from both sides} \\
y &= -\frac{\sqrt{2}}{2} && \text{Divide both sides by 2} \\
\sin(x) &= -\frac{\sqrt{2}}{2} && \text{Substitute back to the original variable}
\end{aligned}
$$

This step is easily forgotten.

Try it: Solve the equation $4\exp(x) + 6 = \ln(4)$ for $\exp(x)$. Do it once using a substitution similar to the example above, and then do it without that substitution. Use a complete presentation both times.

10.1 Reading Mathematical Expressions - Worksheet 1

1 Consider the expression $(x+2)(x-5)$. Describe the "big picture" perspective of the expression and put boxes around the terms as appropriate.

2 Consider the expression $x+2(x-5)$. Describe the "big picture" perspective of the expression and put boxes around the terms as appropriate.

Many students treat this calculation like the previous one. It's an easy "gotcha" problem to catch students that aren't paying attention.

3 Determine whether you think the following equation is valid. Explain your reasoning.

$$\sin(x+y) = \sin(x) + \sin(y)$$

Hint: This is one of the special named functions that was discussed in this chapter.

4 Solve the equation $x\ln(2) + 3 = -4$. Do it once using a substitution for $\ln(2)$, then do it without a that substitution. Use a complete presentation both times.

10.2 Reading Mathematical Expressions - Worksheet 2

1 Consider the expression $2x(x-3)+4$. Describe the "big picture" perspective of the expression and put boxes around the terms as appropriate.

2 Evaluate the expression $2x(x-3)-5$ when $x=2$. Use a complete presentation.

There are no examples of this! Use your best judgment of what a "complete presentation" would like for this calculation based on your experience. Start trusting yourself.

3 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\begin{aligned} 3x + f(4) &= f(10) & \\ 3x + 4 &= 10 & &\text{Cancel out the } f \\ 3x &= 6 & &\text{Subtract } 4 \text{ from both sides} \\ x &= 2 & &\text{Divide both sides by } 3 \end{aligned}$$

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\begin{aligned} \exp(x) + 3 &= 8 & \\ \exp(x) &= 5 & &\text{Subtract 3 from both sides} \\ x &= \frac{5}{\exp} & &\text{Divide both sides by } \exp \end{aligned}$$

10.3 Reading Mathematical Expressions - Worksheet 3

1 Consider the expression $(x+1)^2 - (x-1)^2$. Describe the "big picture" perspective of the expression and put boxes around the terms as appropriate.

2 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\begin{aligned}
(x+4)^2 - (x-3)^2 &= (x^2+16) - (x^2-9) && \text{Distribute the square} \\
&= x^2 - x^2 + 16 + 9 && \text{Rearrange the terms} \\
&= 25 && \text{Combine like terms}
\end{aligned}$$

3 Simplify the expression $(x+1)^2 - (x-1)^2$ using a complete presentation.

4 Solve the equation $2\tan(x) - 5 = -3$ for $\tan(x)$. Do it once using a substitution for $\tan(x)$, then do it without that substitution. Use a complete presentation both times.

10.4 Reading Mathematical Expressions - Worksheet 4

1 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$6\exp(2) = 10$$

$$x = \frac{10}{\exp(2)} \qquad \text{Divide both sides by } \exp(2)$$

$$x = \frac{5}{\exp(1)} \qquad \text{Reduce}$$

2 Solve the equation $ax + b = c$ for the variable x using a complete presentation.

3 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$x\log(5) = \log(7)$$

$$x = \frac{\log(7)}{\log(5)} \qquad \text{Divide both sides by } \log(5)$$

$$x = \frac{7}{5} \qquad \text{Cancel out the } \log$$

4 Solve the equation $2\log(x) = \log(x) + 5$ for $\log(x)$. Do it once using a substitution for $\log(x)$, then do it without that substitution. Use a complete presentation both times.

Students can start to feel lost with a problem like this. But the substitution helps them to visually identify where the variable terms are located, and it helps them to see how to proceed.

10.5 Reading Mathematical Expressions - Worksheet 5

1 Solve the equation $ax + b = cx + d$ for the variable x using a complete presentation.

2 Solve the equation $x\sin(1) + \cos(2) = x\ln(3) - f(4)$. Use a complete presentation.

Try to do this without making substitutions.

3 Solve the equation $3\ln(x) + \ln(4) = 8$ for $\ln(x)$. Use a complete presentation.

4 Solve the equation $3x + \log(6) = \exp(3)$. Use a complete presentation.

10.6 Deliberate Practice: Solving for Variables and Function Expressions

Focus on these skills:

- Write the original expression.
- You do not need to use a substitution to replace the function expressions, but it might help.
- Avoid inappropriate cancellations and other algebraic errors involving functions.
- Present your work legibly.

Instructions: Solve for the indicated quantity.

1. $\ln(3)x + 5 = \ln(2)x - 7$ for x

2. $\pi x + \sqrt{3} = 2x + \cos(2)$ for x

3. $\exp(4)x + 3 = \exp(2)x + \exp(5)$ for x

4. $\sqrt{5}x - \log(4) = \pi x + \sqrt{7}$ for x

5. $2\sin(x) + \sqrt{3} = 4\sin(x)$ for $\sin(x)$

6. $5\cos(x) + \sqrt{2} = 3\cos(x) - 2$ for $\cos(x)$

7. $4\ln(x) + 3 = -2\ln(x) + \ln(4)$ for $\ln(x)$

8. $5\exp(x) - \sqrt{6} = 2\exp(x) - \cos(1)$ for $\exp(x)$

9. $3f(x) - 4f(2) = g(3) + 6$ for $f(x)$

10. $5\sqrt{x} + \ln(3) = 4\exp(2) - 3$ for $\sqrt{x}$

10.7 Closing Ideas

There's an internet math meme that makes its way around social media every now and then. There are a few versions of it, but they all look something like this:

$$\text{Calculate: } 6 \div 2(1+2)$$

People will argue for hours over whether the correct answer is 1 or 6. Some people will insist on the "rule" that division and multiplication and done left-to-right, but others will note that the implied multiplication is usually given priority. For example, in writing $5 \div 2x$, there's virtually no context in which we would separate the 2 from the x.

But all of that arguing misses the point. The "answer" is that the person writing the problem did not communicate effectively. The person who wrote the expression did not use notation in a clear manner. And that's an important lesson. Mathematics is not about arcane rules, but about communicating ideas.

One of the ongoing challenges for math courses is that students can keep advancing forward with a weak foundation. They can often do well enough to pass the new material while there are weaknesses from previous sections that remain unaddressed. And this can continue for a while, sometimes multiple years, until there's a certain moment where things suddenly make no sense at all.

Sometimes, students will get to that spot and try to brute force their way through it. It is not often successful from the educational perspective, though with enough effort students they may still be able to "pass" the material (even if just barely).

The first part of this book has been focused on shoring up a number of key algebraic ideas and concepts. This last section is an important step towards true algebraic fluency. As soon as you are able to "see through" all of the notation and break down complicated expressions to their more basic components, you have access to a much wider range of algebraic thought processes. Going back to the language analogy, this is the point where you cross over from speaking in broken sentence fragments into the early stages of fluency.

The next major step in forward development is contextualized practice. It's not enough to simply see these algebraic ideas in isolation. They need to be brought into perspective through the lens of other mathematical ideas. And that is where the college level math course will take over.

Study tip: Keep a log of your basic algebra errors. Write down (using a complete presentation!) what you did and what you were supposed to do. This will help your brain catch the patterns and fix them. This is one of the fastest ways to correct these types of errors. After the third or fourth time, you will have trained your brain to watch for that error.

The rest of this book is about backwards development. In other words, it's about finding and filling some of the earlier holes that create problems for students. In a real sense, we will be working backwards from concepts found in high school algebra down to elementary school arithmetic. The content is full of a number of simple ideas that are sometimes missed by students as they're coming up through the educational system. And those are sometimes the concepts that help to support higher levels of mathematical thinking.

10.8 Going Deeper: Your Calculator and You

Teachers used to say that you wouldn't be carrying around a calculator in your pocket all the time. They were wrong. We do this all the time. But it's surprising that even though we have these calculators, most people can't actually use them properly. Most people can use them for simple arithmetic, but will make errors for even slightly complex situations.

From what I can tell, the basic challenge is that calculators behave in a way that doesn't always match up with how students think the calculator should behave. Here is an example:

$$\frac{2+6}{3+1}$$

When students read this in their minds, they see it as "2 plus 6 divided by 3 plus 2" and proceed to press the following buttons:

$$\boxed{2}\ \boxed{+}\ \boxed{6}\ \boxed{\div}\ \boxed{3}\ \boxed{+}\ \boxed{1}\ \boxed{=}$$

If you do the calculation by hand, you can see that the result will be 2. But the calculator will give the answer 5. Most students won't even recognize that there's something wrong because they have learned to trust their calculators. The problem here is exactly the same as the problem with the $x^2 + 3/x + 2$ example from earlier in this section. The calculator is trying to apply the order of operations based on what you've entered, but that results in the calculation of something other than what you had intended.

Most modern calculators will give the wrong answer of 5. But some will give a wrong answer of 3.33333333. Those calculators are evaluating everything step-by-step and not even trying to manage the order of operations for you.

The problems only get worse as you get further along in math. There are other situations where different calculators are programmed to interpret expressions differently. A common test for calculators is to see how it handles implied multiplication. Consider the following key presses:

$$\boxed{1}\ \boxed{\div}\ \boxed{2}\ \boxed{\sqrt{\ }}\ \boxed{2}\ \boxed{=}$$

Some calculators will interpret this as $\frac{1}{2\sqrt{2}}$ (giving 0.3536), others will treat this as $\frac{1}{2}\sqrt{2}$ (giving 0.7071), and still there's a whole other set of calculators that might just give you the number 0.5.

The calculators that give 0.5 are calculators that evaluate the square root when you push the button, as opposed to letting you type out the full expression. There's actually something going on with the memory when you press the square root button. Basically, the second 2 is overwriting the $\sqrt{2}$ calculation, so that you end up with just $1 \div 2$.

Your instinct may be to ask the following question: "Which one is right?" And the reason you might do this is because you're used to there always being a right answer. But that's simply not the case. Just as with the math meme in the closing idea, this isn't a problem of right and wrong. It's a problem of communication. Did the buttons communicate the right instruction to the calculator? In this case, you don't know what the intention was, and neither does the calculator. It's just blindly following its instructions.

Fortunately, most modern calculators give you a way around this problem. They give you parentheses so that you can be explicit with the calculator what you want it to do. And this is a habit that you will want to develop for any time you use a computational device to perform a calculation. It also doesn't hurt to be explicit about multiplication calculations.

The problems only get worse when you start using functions. Some calculators evaluate the function when you press the button, and others wait for you to type in your full expression before it decides what it wants to calculate. And depending on which type of calculator you have, you'll need to do things in different orders to get the result that you want. If you didn't read the margin comment about the square root function, you should read that now.

When you get into trigonometry, you will also need to know whether your calculator is in degree mode or radian mode. And failing to switch to the appropriate mode will lead to even more errors. Basically, you're picking between different units for numbers. It's like if someone were to tell you that the length of an object is 3. On its own, you don't know what units of measurement they used, and so it's impossible to know if they meant 3 inches or 3 miles.

Ultimately, it's impossible to give clear guidance for how your specific calculator will behave, because there's so much variation between them. You just need to know how to use it right. There are a couple techniques you can use. The first was mentioned earlier, which is to use parentheses to explicitly tell the calculator the order you want it to calculate everything. Another technique is to break the calculation down into smaller steps, so that you're only putting in one calculation at a time, and then writing down that number on your paper before going to the next calculation. This is especially helpful if you have a calculator that evaluates functions immediately. This saves you from the mental gymnastics of trying to plan ahead to make sure that you've worked out the correct order for typing everything into your calculator.

In some ways, even if your calculator has parentheses it's preferable for you to do it step by step and write out the results of each set of calculations. The reason is that this is easier to work with when you check your work. For longer expressions, you can end up with four or five sets of parentheses, and it can be a lot of work to make sure they're all correct. It also helps you to mentally think through the steps of the calculation to make sure that you really understand what you're doing. But the precise bounds on when to switch is going to depend on the amount of practice you have with your calculator.

10.9 Solutions to the "Try It" Examples

1

$$\boxed{\frac{x+3}{x-2}} + \boxed{3x}$$

This is the sum of a fraction and a monomial.

If you also put boxes around the numerator and the denominator, that would also be acceptable. But the "big picture" is the sum of two terms.

2

$$
\begin{aligned}
x\sin(1) + 5 &= 3x - \cos(2) && \\
x \cdot \boxed{A} + 5 &= 3x - \boxed{B} && \text{Substitute} \\
x \cdot \boxed{A} &= 3x - \boxed{B} - 5 && \text{Subtract 5 from both sides} \\
x \cdot \boxed{A} - 3x &= -\boxed{B} - 5 && \text{Subtract } 3x \text{ from both sides} \\
\left(\boxed{A} - 3\right) x &= -\boxed{B} - 5 && \text{Factor out the } x \\
x &= \frac{-\boxed{B} - 5}{\boxed{A} - 3} && \text{Divide by } \boxed{A} - 3 \\
x &= \frac{-\cos(2) - 5}{\sin(1) - 3} && \text{Substitute}
\end{aligned}
$$

$$
\begin{aligned}
x\sin(1) + 5 &= 3x - \cos(2) && \\
x\sin(1) &= 3x - \cos(2) - 5 && \text{Subtract 5 from both sides} \\
x\sin(1) - 3x &= -\cos(2) - 5 && \text{Subtract } 3x \text{ from both sides} \\
(\sin(1) - 3)\, x &= -\cos(2) - 5 && \text{Factor out the } x \\
x &= \frac{-\cos(2) - 5}{\sin(1) - 3} && \text{Divide by } \sin(1) - 3
\end{aligned}
$$

3

$$
\begin{aligned}
4\exp(x) + 6 &= \ln(4) && \\
4y + 6 &= \ln(4) && \text{Substitute } y = \exp(x) \\
4y &= \ln(4) - 6 && \text{Subtract 6 from both sides} \\
y &= \frac{\ln(4) - 6}{4} && \text{Divide both sides by 4} \\
\exp(x) &= \frac{\ln(4) - 6}{4} && \text{Substitute back to the original variable}
\end{aligned}
$$

$$
\begin{aligned}
4\exp(x) + 6 &= \ln(4) && \\
4\exp(x) &= \ln(4) - 6 && \text{Subtract 6 from both sides} \\
\exp(x) &= \frac{\ln(4) - 6}{4} && \text{Divide both sides by 4}
\end{aligned}
$$

A Pause for Reflection

Congratulations! You have just completed the "main trunk" of the course. The algebraic skills that we have been discussing are the foundation of college level mathematics. But that doesn't mean that there's nothing more to learn. Mathematical knowledge is deeply scaffolded. And weaknesses in the lower level can manifest in an unstable understanding down the line. What we have done up to this point is attempt to plug the biggest holes as fast as possible so that you can move forward in your college level math courses. It's really just a patchwork job that tries to get you to a place where you have a decent chance of being successful at that college level coursework.

When we say that knowledge is scaffolded, we mean that new ideas are built on top of old ones.

The sections that follow are going to be focused on getting deeper into the foundation in search of filling the cracks that lie beneath the surface. As we follow the different "branches" of mathematical thinking, the hope is that you will come to a new and deeper understanding of ideas that you have seen for years. Some of these ideas may help feed directly into your college level work, but other things may simply be about uncovering a new perspective of mathematics.

These are the branches:

- The coordinate plane and linear equations
- Fractions and decimals
- Arithmetic
- Applications

As we have been emphasizing throughout this text, there is more to math than just executing calculations. Yes, that's a part of it, but there's an element of communicating and understanding the ideas and a curiosity for how ideas fit together. Ultimately, this is all about building a mindset towards mathematical thinking that is robust enough to help you be successful at whatever you do. One of the reasons that mathematical skills are so highly valued is not because there aren't a lot of people who can do the raw manipulations (we have computers to do that for us), but it's because there aren't a lot of people who can meaningfully interpret that information and apply it to new situations.

For these reasons, it's important to take a pause every now and then and think about what's happening beyond just the calculations and think about what's happening to your mindset as you've been working your way through the content.

Metacognition is the understanding and awareness of your own thinking.

In this portion of the course, we have covered the following topics:

- Basic Algebraic Presentation
- Variables in Expressions and Equations
- Like and Unlike Terms
- Simplifying Expressions and Solving Equations
- Variables and Substitutions
- The Properties of Exponents
- Products of Polynomials
- Common Factors
- Factoring Quadratic Polynomials
- Reading Mathematical Expressions

Questions About the Content

1 Were there any topics that you had seen before, but you understand better as a result of working through it again?

2 Were there any ideas that you had never seen before?

3 Based on your experience, which of these ideas seems the most important to understand well?

4 Did any part of the presentation make you curious about math in a way that went beyond the material? Are there questions or ideas that you would like to explore?

Examples:

- Why can't we divide by zero?
- Why is multiplication commutative?

Questions About You

1 How has your mathematical writing evolved from the beginning of the course? Do you find yourself thinking in different ways?

2 Did you have any “Aha!” moments where you had an insight into something that you had not noticed before?

3 What is the biggest mathematical connection that you made?

Mathematicians sometimes think of ideas that fit perfectly together as being “beautiful.”

4 What is your level of confidence in mathematics? Is this higher, lower, or about the same as it was at the beginning of class?

Get it Straight: Lines and the Coordinate Plane

Learning Objectives:

- ☐ Identify and determine solutions to linear equations.
- ☐ Locate points on the coordinate plane.
- ☐ Sketch solutions to linear equations by plotting points, including horizontal and vertical lines.

In Definition 4.4, we saw that solving an equation means to find the value (or values) of the variable (or variables) that make the equation true, and that solutions are the specific values of the variables that accomplish that. We are going to take another look at those ideas, but in the context of using multiple variables simultaneously. Specifically, we are going to be working with linear equations in the variables x and y.

All of the ideas will work with other variable combinations, but the variables x and y are the most common.

Definition 12.1. A *linear equation in the variables x and y* is an equation that is equivalent to one of the form $ax + by = c$ for some constants a, b, and c. This form is known as the *standard form* of the equation.

Two equations are equivalent if there is a sequence of valid algebraic manipulations to convert one into the other and back again.

1

For example, the equation $3x + 2y = 7$ is true when $x = 1$ and $y = 2$. We can represent this using the notation $(1, 2)$. This is known as an ordered pair because the order of the numbers matters. The first value inside the parentheses is called the x-coordinate, and the second value inside the parentheses is called the y-coordinate. We can write this using symbols as $(x, y) = (1, 2)$.

Notice that this is not the only solution. Here are some others: $(-1, 5)$, $(3, -1)$, and $(0, 3.5)$. In fact there are infinitely many solutions.

Try it: Determine four solutions to the equation $4x - 3y = -1$, including at least one solution with a negative value and one solution that uses at least one decimal or fraction.

Unless otherwise specified, we assume that an ordered pair is being represented by the variables x and y, in that order.

Note: Do not worry about presenting your work here. You ought to be able to find values mentally.

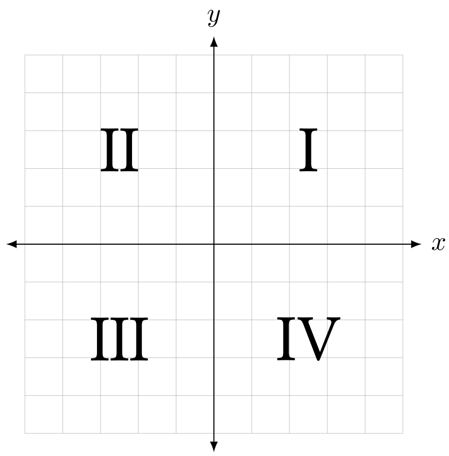
Long lists of ordered pairs are not the most intuitive way to present solutions. For more than just a couple points, it makes sense to transition to using a chart, such as the one in the margin. Sometimes a chart like this can help us see a pattern in the numbers if the pattern is simple. But even with that, charts have limited value because it's still just a list of numbers. It would be better to use a more visual representation of this information.

Mathematicians often use a coordinate plane to represent solutions to two variable equations. The coordinate plane is a picture where specific positions represent specific ordered pairs. Most students are familiar with the basic design of the standard rectangular coordinate grid, which is the grid we will be using.

Solutions of $x + y = 5$

x	y
0	5
1.5	3.5
4	1
6	-1

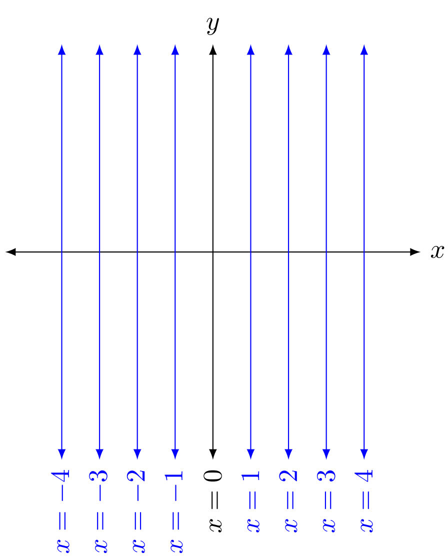
There are other coordinate systems, but they all follow a similar conceptual construction.

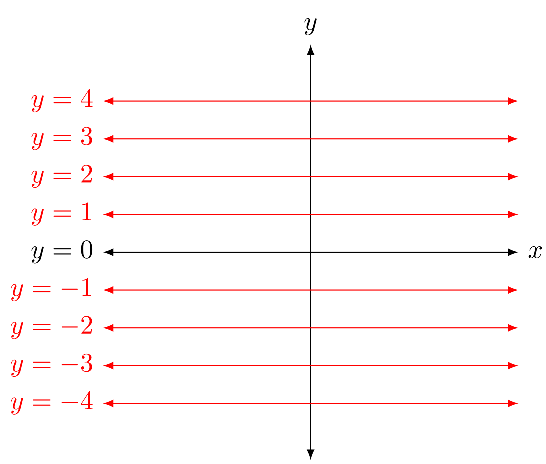


Here is a quick reminder of some of the basic terminology:

- The labeled horizontal line is called the x-axis.
- The labeled vertical line is called the y-axis.
- The intersection of those two lines is called the origin.
- The quadrants are numbered with capital roman numerals, starting with I on the upper right and working around in a counter-clockwise manner.

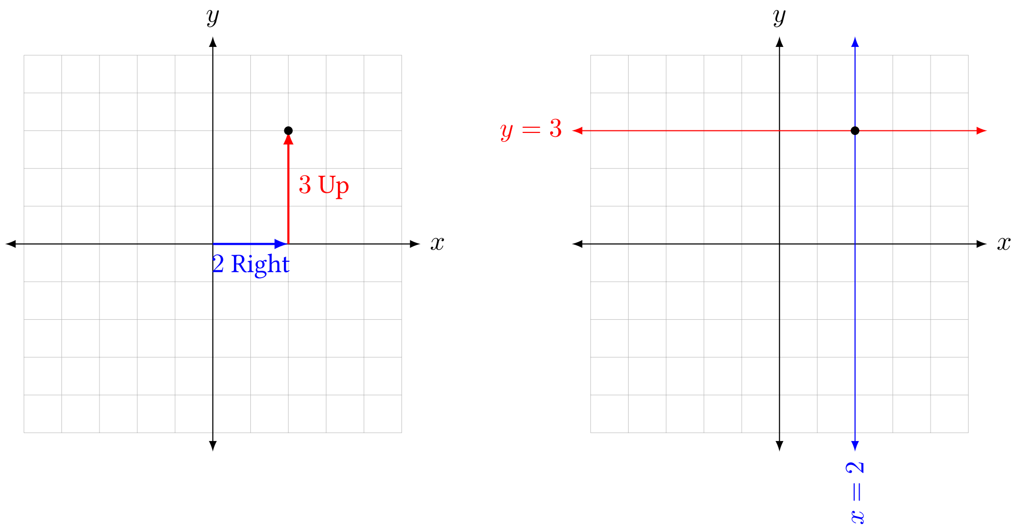
Students often learn coordinates as a sequence of motions starting from the origin. The point $(2,3)$ is located by starting from the origin, moving right 2, and then moving up 3. Negative x-coordinates correspond to moving to the left instead of to the right, and negative y-coordinates correspond to moving down instead of up.





But there is another way to look at this which is a little bit more general. Rather than thinking about this in terms of movement, we can think about this in terms of the intersection of two lines. Lines of the form $x =$ (Number) are vertical lines corresponding to the coordinates on the x-axis, and lines of the form $y =$ (Number) are horizontal lines corresponding to the coordinates on the y-axis. It is the overlap of these lines that creates the coordinate grid.

Once you have this, then you can see that the positions are actually the intersection of two of these lines, corresponding to the specific x value and the specific y value. Here are the two ways of interpreting the point $(x, y) = (2, 3)$ visualized side-by-side.



2 Part of mathematical thinking is the ability to conceptualize the same result in multiple ways. Locating points on a coordinate grid is an example of this. We can think of it both in terms of movement and in terms of the intersection of lines.

Try it: Plot the point $(-3, 1)$ and draw a visualization for both conceptualizations of locating the point.

We can generalize the idea of giving locations as the intersection of two lines by allowing ourselves to use curves. For example, locations on the earth are found as the intersection of the latitude and longitude lines (which are actually curves on the globe). Your location in a city can often be described as being near the intersection of two streets (which may not be straight). In trigonometry, there's another coordinate grid that's built around circles and lines pointing out from the origin called polar coordinates.

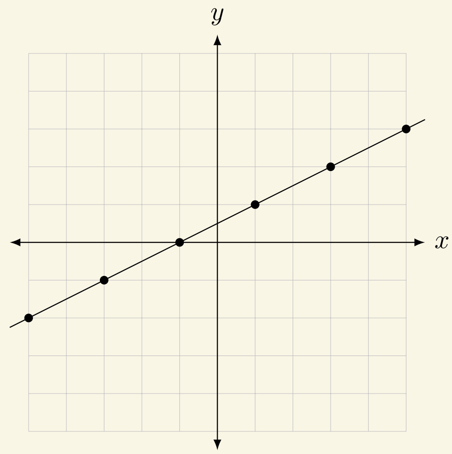
All of these ideas can extend into three or more dimensions.

Once we have the ability to locate points on the coordinate plane, we can then plot lots of points on the same coordinate grid and look for a pattern.

3 We saw earlier that we can generate solutions to a linear equation by inspection. If we plot those solutions on a grid, they will all appear in a line, which is why we call the equation a linear equation. Here is an example of solving the equation $x - 2y = -1$. After plotting the points, we can draw in the shape that is implied by the points.

When mathematicians say "by inspection" they mean that you can look at the problem and work it out in your head.

x	y	(x, y)
-5	-2	$(-5, -2)$
-3	-1	$(-3, -1)$
-1	0	$(-1, 0)$
1	1	$(1, 1)$
3	2	$(3, 2)$
5	3	$(5, 3)$



Try it: Find four solutions of the equation $3x - 2y = -1$. Plot the points and sketch the solution.

An important feature to recognize is that the line that is drawn represents all of the solutions. It turns out that every single point on the line will solve the equation, even the ones that fall in between the grid points. This is where all those decimal solutions can be found.

However, while the sketch of the line is an important tool for building intuition, you need to be very careful about trying to guess exact values on the basis of a sketch. If you're reading values from a graph that are not on the grid lines, you must always acknowledge that you are only giving an approximation.

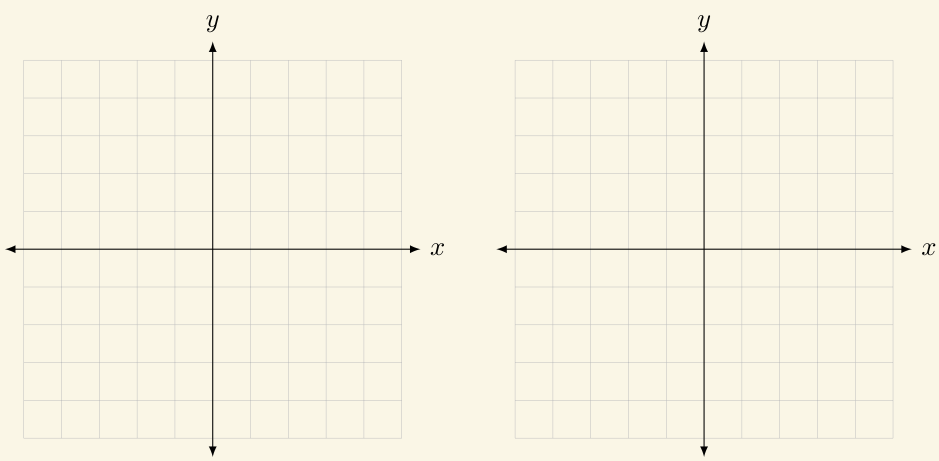
Can you really tell the difference between 0.5 and 0.49 by looking at a picture?

12.1 Lines and the Coordinate Plane - Worksheet 1

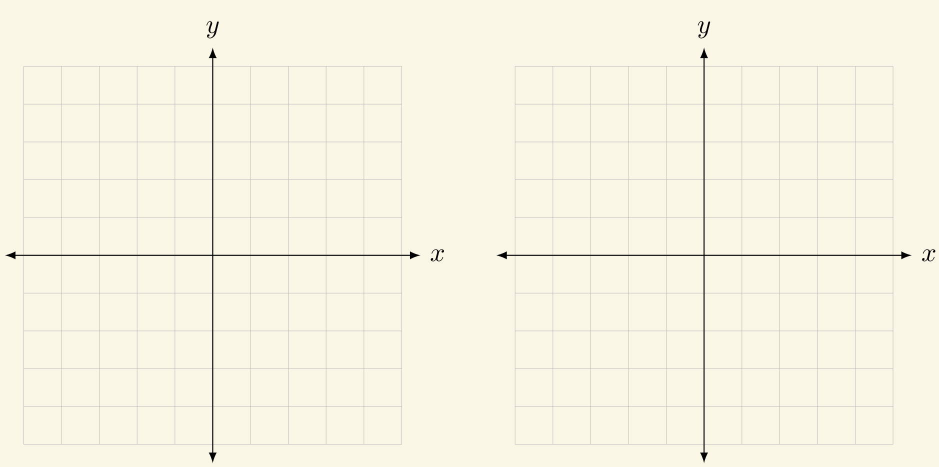
1 Determine four solutions of the equation $2x - 5y = 2$, including at least one solution with a negative value and one solution that uses decimals or fractions.

Whenever you write points as ordered pairs, you must write the parentheses. Don't be lazy!

2 Plot the point $(4, 2)$ and draw a visualization for both conceptualizations of locating that point.



3 Plot the point $(-3, -4)$ and draw a visualization for both conceptualizations of locating that point.

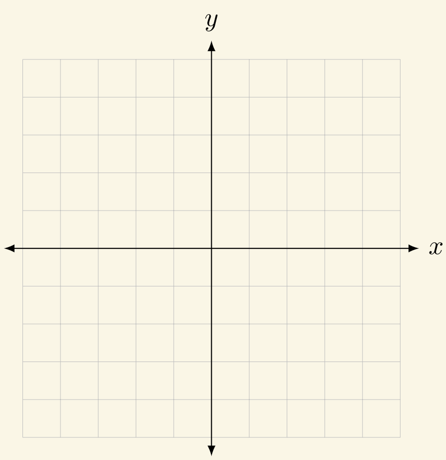


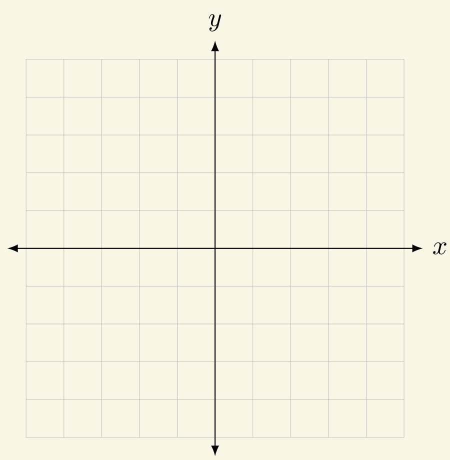
12.2 Lines and the Coordinate Plane - Worksheet 2

1 Determine 4 solutions of the equation $x - 3y = -2$, including at least one solution with a negative value and one solution that uses decimals or fractions.

2 Plot the point $(0, 3)$ and draw a visualization for both conceptualizations of locating that point.

Zeros throw students off for some reason.

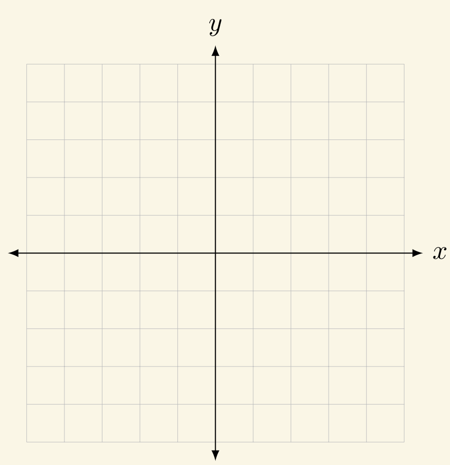




3 Find four solutions of the equation $2x - 3y = 1$. Plot the points and sketch the solution.

Try to pick points that fit on the given coordinate grid when plotting points. You will sometimes need to use off-grid points, but you should try to avoid that because the plots become increasingly inaccurate when you do.

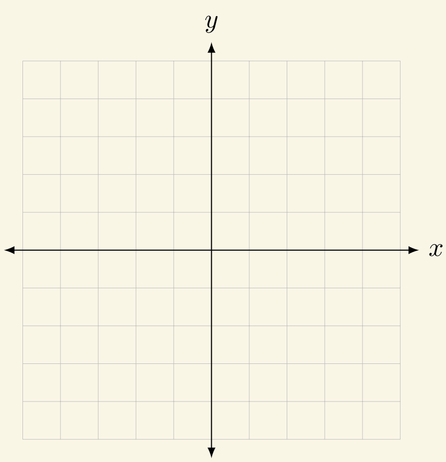
x	y	(x, y)

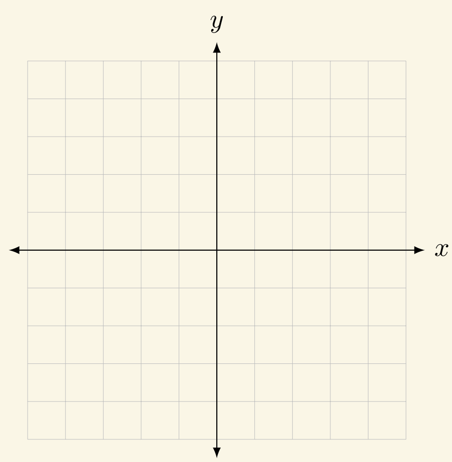


12.3 Lines and the Coordinate Plane - Worksheet 3

1 Determine 4 solutions of the equation $2x - 3y = -5$, including at least one solution with a negative value and one solution that uses decimals or fractions.

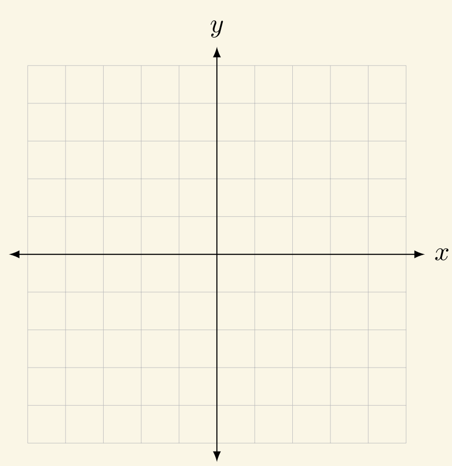
2 Plot the point $(-5, 0)$ and draw a visualization for both conceptualizations of locating that point.





3 Find four solutions of the equation $-x + y = 3$. Plot the points and sketch the solution.

x	y	(x, y)



12.4 Lines and the Coordinate Plane - Worksheet 4

1 Each chart represents some solutions of a linear equation, but the equation of that linear equation isn't given. Determine three more points on the line based on the existing solutions.

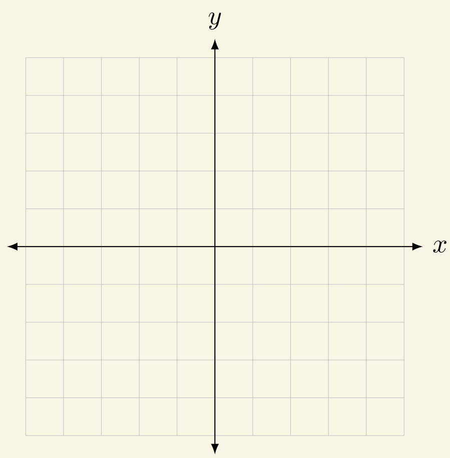
x	y	(x, y)
1	3	$(1, 3)$
2	1	$(2, 1)$
3	-1	$(3, -1)$

x	y	(x, y)
-4	5	$(-4, 5)$
-2	4	$(-2, 4)$
0	3	$(0, 3)$

This is something of a puzzle that is built on your experience. If you're really stuck, try sketching the graph and looking for a pattern.

2 Find four solutions of the equation $-3x - y = 4$. Plot the points and sketch the solution.

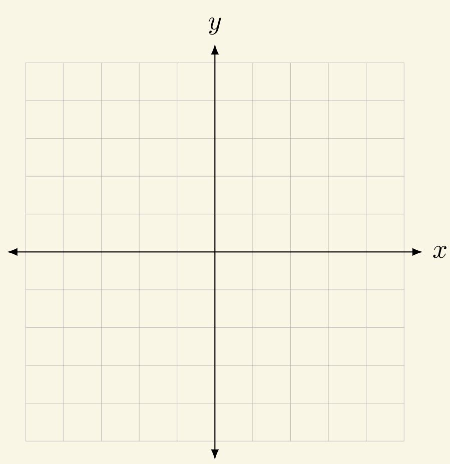
x	y	(x, y)



3 Find four solutions of the equation $x = -3$. Plot the points and sketch the solution.

When there is no restriction on the y value, it means that you can pick the y-coordinate to be anything you want.

x	y	(x, y)



12.5 Lines and the Coordinate Plane - Worksheet 5

1 Each chart represents some solutions of a linear equation, but the equation of that linear equation isn't given. Determine three more points on the line between the given points.

x	y	(x, y)
-3	-2	$(-3, 2)$
4	5	$(4, 5)$

x	y	(x, y)
5	-3	$(5, -3)$
-5	2	$(-5, 2)$

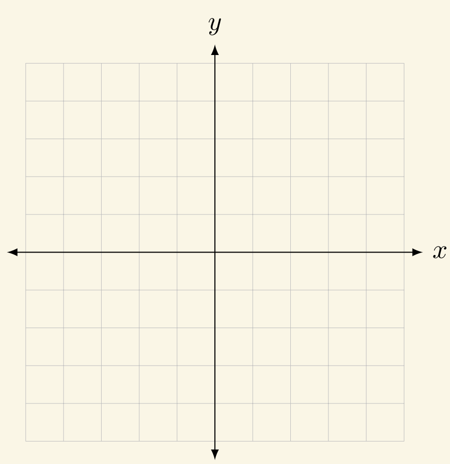
2 Each chart represents some solutions of a linear equation, but the equation of that linear equation isn't given. Determine the missing coordinates based on the given points.

x	y	(x, y)
-3	4	$(-3, 4)$
-2	2	$(-2, 2)$
0		
	-4	
3		

x	y	(x, y)
5	3	$(5, 3)$
2	1	$(2, 1)$
-4		
	7	
11		

3 Find four solutions of the equation $4x - 3y = -3$. Plot the points and sketch the solution.

x	y	(x, y)



12.6 Deliberate Practice: Graphing Lines

Focus on these skills:

- Write the original equation.
- Pick integer coordinate points. The problems are written to work with a grid that ranges from -5 to 5 in both axes, but you may use a larger grid if you want.
- Grids are not included. You can either buy a pad of graph paper or use printable graph paper from internet (which you should be able to find for free).
- Present your work legibly.

Please don't freehand your graph paper.

Instructions: Find four solutions of the equation, then plot the points and sketch the solution.

1 $x + y = 3$

2 $-2x + y = 1$

3 $x - y = 0$

4 $x + 2y = 3$

5 $3x - 2y = 1$

6 $-x + 3y = 2$

7 $x - 2y = 0$

8 $2x - 3y = -1$

9 $3x - y = 1$

10 $2x + 4y = 4$

12.7 Closing Ideas

This section focused more on developing your intuition for lines than on the algebra. Linear equations are extremely common in practical applications. For example, if a burger costs $3 and you want 5 burgers to bring to your friends, how much will that cost? This can be modeled as a linear equation.

The algebra is also important. We'll see more of that in the next section.

Of course, simple cases such as this can be done with simple methods. As you start to work with "real life" situations, it's usually not going to solved by quick mental arithmetic. You're going to need better tools in your toolbox. And this is where having a background in algebraic reasoning and mathematical thinking will help.

As you worked your way through the worksheets, you might have started to notice that moving from one point to another along a line always required your variables to change in tandem. For example, the x-coordinate might increase by 1 every time the y-coordinate decreased by 2. We will see that this is a reflection of the idea of the "slope" of a line, which we will explore more deeply in the next section.

Without telling you, there were two types of modeling that were introduced in the last couple worksheets. These are known as interpolation and extrapolation. The basic idea of interpolation means to fill in data points between existing data points. (See the "Going Deeper" section that follows.) Extrapolation means modeling data beyond the existing data points. Linear equations are often used for models of these types. It's usually more complicated because real life data rarely ever fits exactly on a line, so you have to go through a process of finding the "best" line that matches the data, but that application will have to wait for a science or business course.

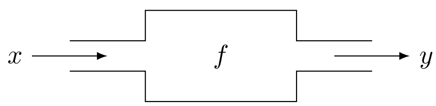
12.8 Going Deeper: Graphing Functions

Most of the time in math courses, we use the variable x for the horizontal axis and y for the vertical axis. This is the "standard" choice for coordinate systems when there is no specific context being applied. But a lot of mathematical ideas are applied in very specific contexts, and so we need to be able to translate our ideas from the generic situation to specific ones.

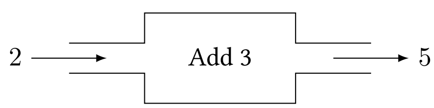
Traditionally, the x variable represents the input (or independent) variable, and y represents the output (or dependent) variable. This language is tied to how mathematicians talk about functions. We will start with the formal definition:

> **Definition 12.2.** A function is a rule that assigns each object of one set to exactly one object a second set.

Conceptually, functions are often thought of as machines. On one side of the machine there's a place for you to put something in, and on the other side there's a place where something comes out.

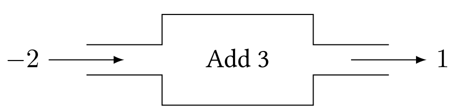


Specifically what comes out will depend on what the machine is designed to do (the "rule" that it has been created to follow). For example, we can have an "Add 3" machine that takes whatever number we give it and add 3 to it.

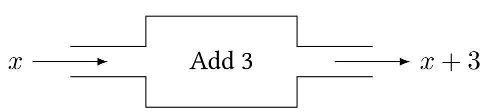


If we picked a different input, we would get a different output.

For some functions, you can get the same output for different input values. For example, if the function squared the number, then both 2 and -2 would have an output of 4.



In fact, if we were to give it a generic quantity, it could give us an expression that represents the appropriate output.

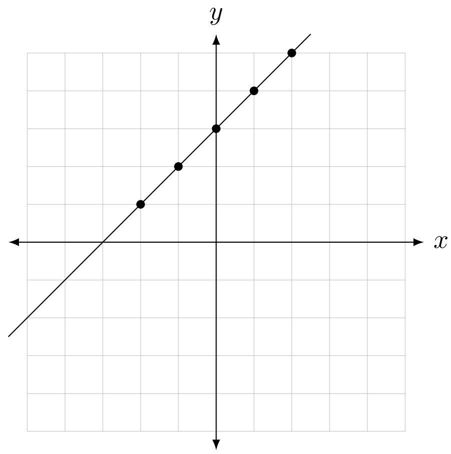


The mathematical shorthand for this concept is $y = f(x)$. This is a versatile notation that we use both to describe the function in general as well as to describe specific input-output pairs of a specific function. And while we often use the letter f to represent a function, we can use other letters or symbols if we wanted. The key observation about this notation is that it defines x as the independent variable and y as the dependent variable. For the "Add 3" example, we would write $y = f(x) = x + 3$.

Many students erroneously read $f(x)$ as "f times x" instead of "f of x." Don't do this! Use the correct language to help your brain build a different mental category for information about functions.

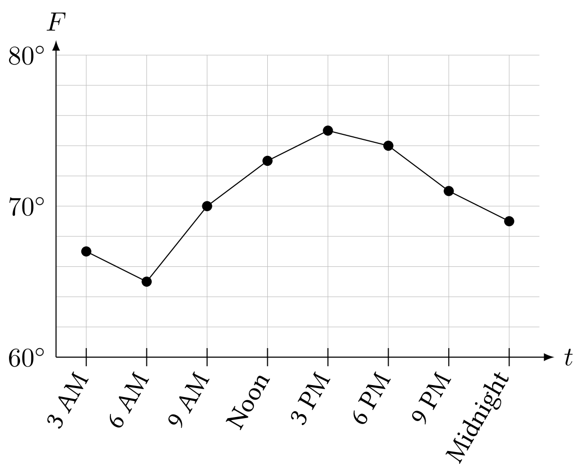
Notice that this is a process that gives us ordered pairs (x, y). If we were to put these values into a chart like the ones we were working with earlier in this section, we could convert this information into a graph.

x	y	(x, y)
-2	1	$(-2, 1)$
-1	2	$(-1, 2)$
0	3	$(0, 3)$
1	4	$(1, 4)$
2	5	$(2, 5)$



There are situations where the data that you have in your table comes from measurements. For example, if you're measuring the temperature throughout the day, you will get a chart of values, but the values probably won't fit a nice formula. Even though there may not be a formula, there's still a meaningful sense that this is a function. The input variable is the time and the output variable is the measurement. And using the same ideas as above, we can put that information into a chart and get a graph of the temperature throughout the day.

t	F
3 AM	67°
6 AM	65°
9 AM	70°
Noon	73°
3 PM	75°
6 PM	74°
9 PM	71°
Midnight	69°



Notice that we had to make some modifications to the graph. We changed the letters to better reflect what we're measuring (t for time and F for degrees Fahrenheit), and we also labeled the axes so that the values could be more easily interpreted. We also altered the time axis to correspond to clock times instead of just having numbers.

But notice how natural this is to read. Given a time, we can get an estimate of the temperature by looking at the graph. This is where a lot of the value of graphing functions starts to come into play. There is a clear relationship between the input value (time) and the output value (temperature). Even if we picked a time that isn't one of the data points, we can still come up with an estimate for the temperature.

You might have noticed that we connected the data points together with straight lines, and that makes the graph look a little artificial. That's almost certainly not how the temperature behaved! It's far more likely to have a smooth shape. This is also why we said that we could *estimate* the temperature instead of saying that we could determine the temperature. There's a little bit of guesswork involved in thinking through the shape of the curve in between the data points.

This is a starting point of the mathematical of interpolation. What types of curves might we try to use to smooth out the graph and get a more realistic shape for the temperature? Unfortunately, the answer to this question is beyond the scope of this book. But that shouldn't stop you from exploring the topic more deeply if you're interested.

Look for "spline interpolation."

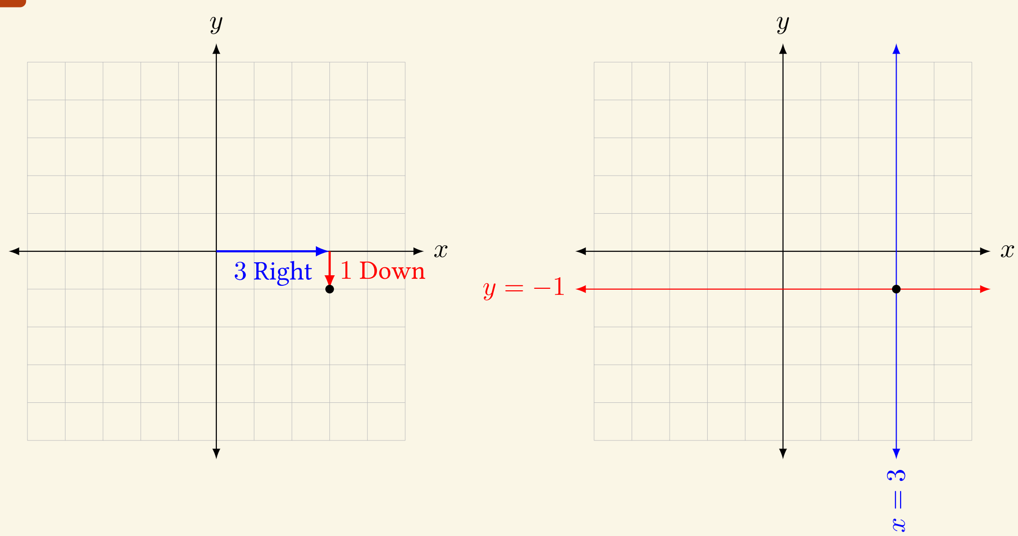
12.9 Solutions to the "Try It" Examples

1 Individual answers may vary.

Integer values:	$(-4,-5),(-1,-1),(2,3),(5,7)$
Fraction values:	$(-\frac{1}{4},0),(0,\frac{1}{3})$
Decimal values (approximations):	$(-0.25,0),(0,0.333)$

"Individual answers may vary" means that there are lots of correct answers and only some are listed. If you picked different numbers, it doesn't mean you're wrong. You just want to check that they make the equation true.

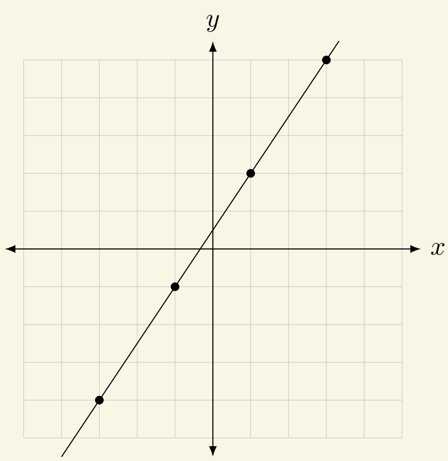
2



3

$$4x - 3y = -3$$

x	y	(x,y)
-3	-4	$(-3,-4)$
-1	-1	$(-1,-1)$
1	2	$(1,2)$
3	5	$(3,5)$



The points may vary, but the line will not.

The Line in the Sand: Slope-Intercept Form

Learning Objectives:

- ☐ Identify the slope of a line from a graph and from a chart of values.
- ☐ Calculate the coordinates of the y-intercept of a line.
- ☐ Convert linear equations into slope-intercept form.
- ☐ Sketch the graph of a line.

In the previous section, we used intuition to find solutions to linear equations in two variables. It is not always the case that we can easily find solutions by inspection. So we are going to develop a more systematic approach to graphing that does not rely on intuition. To do this, we're going to build off of the solutions of lines that we looked at in the previous section.

You might have noticed that when we had charts of values that you can find other charts by changing the x and y values in a consistent manner. This sets up the idea that points on a line satisfy a certain ratio with regards to how the coordinates change. This can be formalized as the concept of the slope of a line.

Definition 13.1. The *slope* of a (non-vertical) line is defined by the following ratio for any two distinct points on the line:

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{The change of } y}{\text{The change of } x}$$

The use of Δ (the capital Greek letter "delta") to represent the change of a variable between two points is also used in physics and chemistry applications.

There is a logical reason for not trying to apply this to vertical lines. In a vertical line, there is no change in the x-coordinate, which means that we would be dividing by zero, and that leads to an undefined expression.

This is not the first time that we have run into the prohibition of dividing by zero. It is something very core to the concept of division. Have you thought through the reason for this yet?

1 The slope can be calculated between any two points on the line and it will result in the same value. The consistent ratio of the changes of the two variables is what makes the line straight.

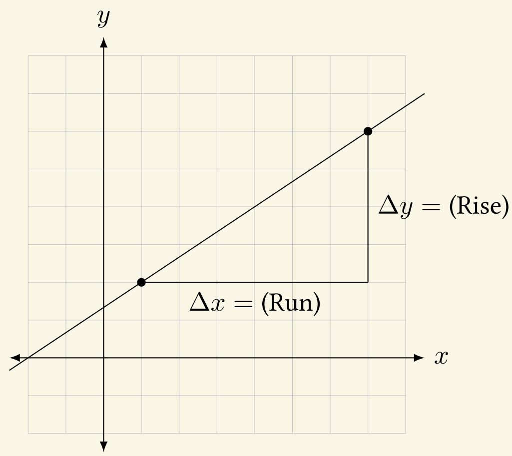
x	y	(x, y)
-5	-2	$(-5, -2)$
-3	-1	$(-3, -1)$
-1	0	$(-1, 0)$
1	1	$(1, 1)$
3	2	$(3, 2)$
5	3	$(5, 3)$

Ask yourself: "What is the change in the x and y coordinates going from this point to that point?"

Try it: Calculate the slope of this line using three different pairs of points. Do not always pick consecutive points for this exercise.

Notice that the fractions reduce. You could actually apply this idea backwards if you wanted to get new points on the line.

2 When working with graphs of lines, we often use a different language to represent the same concept. The "rise" of a function is the change in the y variable between two points (up is positive, down is negative), and the "run" of a function is the change in the x variable between two points (right is positive, left is negative). This leads us to sometimes say "the slope is the rise over the run."



Most people make the run positive because it's more natural to move left-to-right.

Try it: Determine the slope of the line in the diagram above.

Since the fraction reduces, there is a point on the grid between the two marked points. Can you identify it?

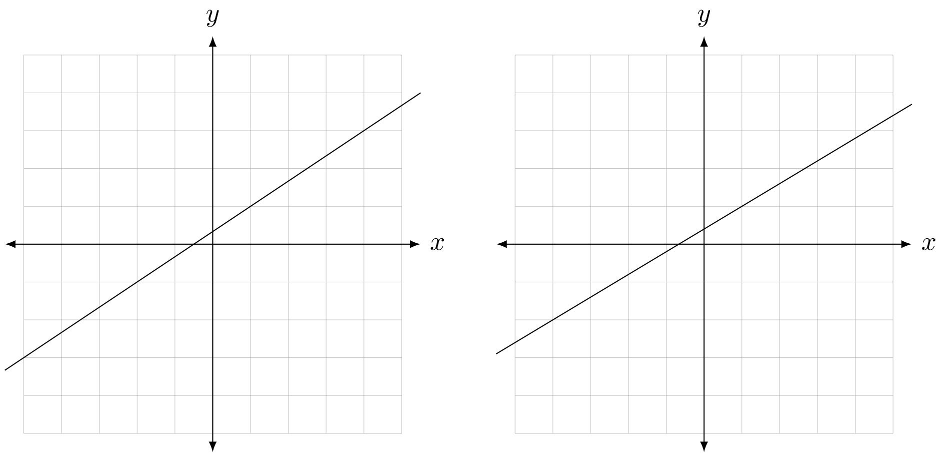
If the slope is positive, the line points from the lower-left to upper-right, and if the slope is negative, the line points from upper-left to lower-right. If the line is horizontal, then the slope is 0.

There is a formula for the slope of the line if you have the coordinates of two points. The formula is commonly given to students, but it can sometimes be a distraction. If you are able to construct a meaningful picture of the line, you will not need to memorize this formula. There are some practical uses from an algebraic perspective, but it's far better to have an understanding of the concepts than simply trying to apply formulas blindly.

The slope of the line that passes through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

The slope is one of the two parameters in the slope-intercept form of a line. The other parameter is the y-coordinate of the y-intercept. The y-intercept is the point where the line crosses the y-axis. When this point is not a point on the grid, students often try to estimate the value. This is somewhat acceptable when sketching a graph, but it is usually not acceptable when reading a graph. The challenge is that is can be very difficult to identify them correctly.

Try to estimate the y-coordinate of the y-intercept in the two graphs below, and you'll understand why estimation is not a reliable technique.



The y-coordinate of the y-intercept of the graph on the left is $\frac{1}{3}$ and the y-coordinate of the y-intercept of the graph on the right is $\frac{2}{5}$.

This highlights the importance of developing an algebraic method for working with lines so that we can avoid needing to make estimates. Specifically, if we're given the equation of a line, it is a common goal to write it in slope-intercept form.

Definition 13.2. The *slope-intercept form* of a line is $y = mx+b$. In this form, m represents the slope of the line and b represents the y-coordinate of the y-intercept of the line.

The reason this works is because the y-intercept of a line is the solution when $x = 0$. But in this form, setting $x = 0$ makes the first term disappear, and you're left with $y = b$, which tells us that the line must pass through the point $(x, y) = (0, b)$.

3 Writing an equation in slope-intercept form is simply a matter of solving for y in a linear equation.

$$3x + 4y = 8$$

$$4y = -3x + 8 \qquad \text{Subtract } 3x \text{ from both sides}$$

$$y = -\frac{3}{4}x + 2 \qquad \text{Divide both sides by } 4$$

Try it: Write the equation $2x - 5y = 10$ in slope-intercept form.

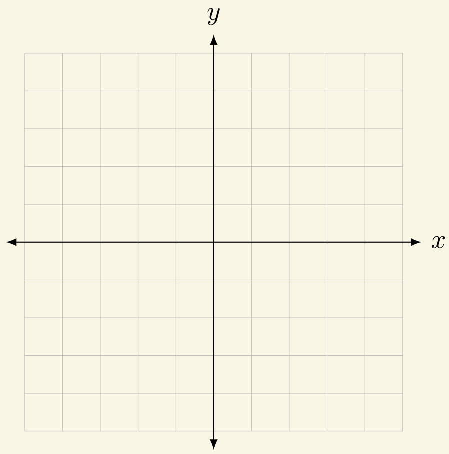
Some people would write the last line as $y = -\frac{3x}{4} + 2$. It doesn't matter which way you do it as long as you recognize that they are both the same result.

Once you have the equation written in slope-intercept form, you have the information to sketch the graph. You can immediately identify the y-intercept, and then you can use the slope to find a second point.

4 The line $y = -\frac{1}{2}x + 1$ has its y-intercept at the point $(0, 1)$. Since the slope is $-\frac{1}{2}$, we can find a second point on the line by moving to the right 2 spaces and down 1 space. This gives us enough information to sketch the line.

Remember: "Slope is rise over run."

You can also move left 2 spaces and up 1 space, but most people move to the right because it feels more natural.



Try it: Graph the line $y = -\frac{1}{2}x + 1$**.**

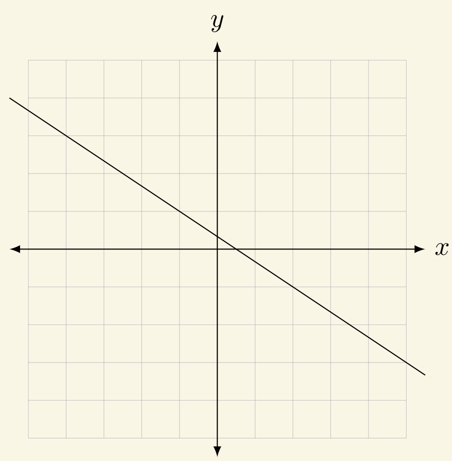
13.1 Slope-Intercept Form - Worksheet 1

1 Calculate the slope of the line that contains the points below using three different pairs of points. Do not always pick consecutive points for this exercise.

x	y	(x, y)
-2	-5	$(-2, -5)$
-1	-3	$(-1, -3)$
0	-1	$(0, -1)$
1	1	$(1, 1)$
2	3	$(2, 3)$

2 Determine the slope of the given line.

Do not estimate coordinates. Find points that are on the grid.

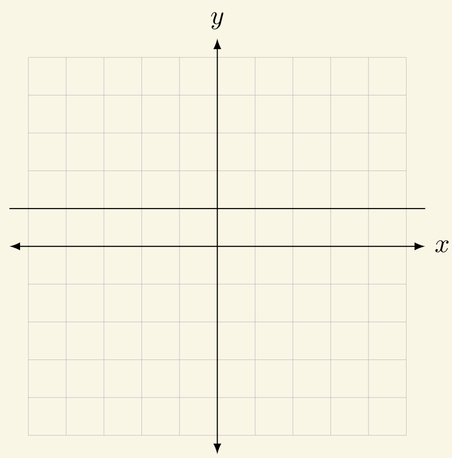


3 Write the equation $-3x + 4y = 12$ in slope-intercept form.

13.2 Slope-Intercept Form - Worksheet 2

1

Determine the slope of the given line.



2

Calculate the slope of the line that passes through the points $(-2, 3)$ and $(4, 0)$.

It may be helpful to make a sketch of the two points.

3

Write the equation $5x - 3y = 10$ in slope-intercept form.

13.3 Slope-Intercept Form - Worksheet 3

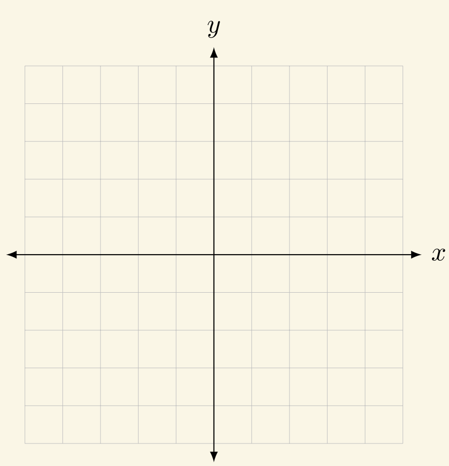
1 Calculate the slope of the line that passes through the points $(2, -1)$ and $(5, 5)$.

2 Calculate the slope of the line that passes through the points $(-2, 0)$ and $(-2, 3)$.

Be careful! Plot the points if you're not sure.

3 Write the equation $-4x - 2y = 8$ in slope-intercept form.

4 Graph the line $y = 2x - 3$.



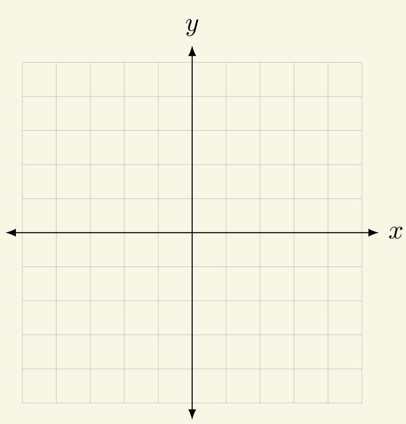
13.4 Slope-Intercept Form - Worksheet 4

1 Calculate the slope of the line that passes through the points $(-1, -1)$ and $(2, -1)$.

2 Write the equation $-3x - 4y = 0$ in slope-intercept form.

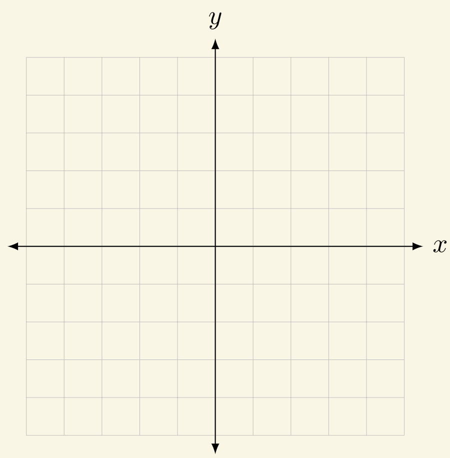
3 Write the equation $-5x + 3y = -7$ in slope-intercept form.

4 Graph the line $y = -\frac{4}{3}x + 2$.

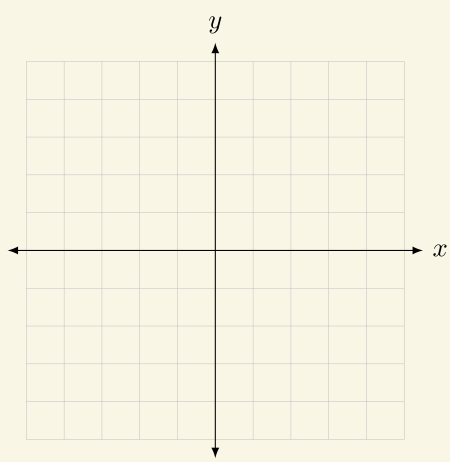


13.5 Slope-Intercept Form - Worksheet 5

1 Graph the line $x - 3y = 6$.



2 Graph the line $-2x + 3y = -4$.



When graphing with fractions, do the best you can in estimating the locations of the points. If possible, use integer coordinate points that are on the line to help make your graphs more accurate.

13.6 Deliberate Practice: Slope-Intercept Form

Focus on these skills:

- Write the original equation.
- Graph the fractional
- Grids are not included. You can either buy a pad of graph paper or use printable graph paper from internet (which you should be able to find for free).
- Present your work legibly.

Please don't freehand your graph paper.

Instructions: Rewrite the equation in slope-intercept form, then graph it.

1. $x + y = -1$
2. $-2x + y = 3$
3. $3x - 2y = 2$
4. $x - 3y = -3$
5. $-x + 2y = -2$
6. $x + 2y = 1$
7. $2x - y = 0$
8. $2x - 3y = -2$
9. $2x + 4y = 3$
10. $3x - 2y = -2$

13.7 Closing Ideas

The slope-intercept form of a line has many useful applications and interpretations. For example:

- The y-intercept can be interpreted as a one-time initiation fee and the slope can be interpreted as a per-use expense. So the cost of an exercise program with a \$15 initiation fee and \$4 per-use fee can be modeled by $y = 4x + 15$, where y is the total cost and x is the number of uses.

- The y-intercept can be interpreted as an initial investment expense (as negative value), and the slope can be interpreted as a per-unit profit for a business model. So if raw materials cost \$100 but each item that can be created sells for \$12 each, then the profit y can be modeled as $y = 12x - 100$, where x represents the number of items sold.

- The slope-intercept form of a line is the standard way that computers present a line of best fit.

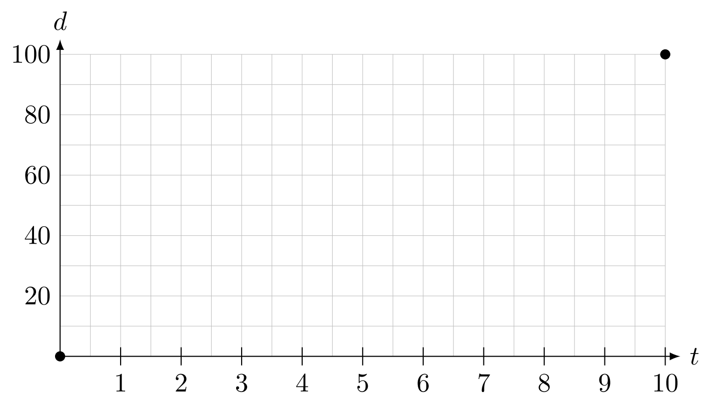
A line of best fit is a used in data analysis as the most ideal approximation of a set of points.

There is a lot more than can be said about the use of lines in applications, but those will have to wait for their corresponding courses. At this point, the goal is that you are comfortable enough with the algebra of two-variable equations to convert them into slope-intercept form, and that you are able to sketch the graphs of such lines.

13.8 Going Deeper: Average Rate of Change

It turns out that the idea of slope is extremely important in both practical and theoretical mathematical thinking. Slopes can be used to represent important relationships between variables, and they one of the core concepts that we see in calculus. So we want to take extra time to really think about how we can understand and interpret the notion of slope.

Let's imagine that we're watching someone run the 100-meter dash, and that it takes them exactly 10 seconds to reach the finish line. It would make perfect sense for us to say that they ran 10 meters per second. But what does that really mean? Let's look at a graph that represents the situation.

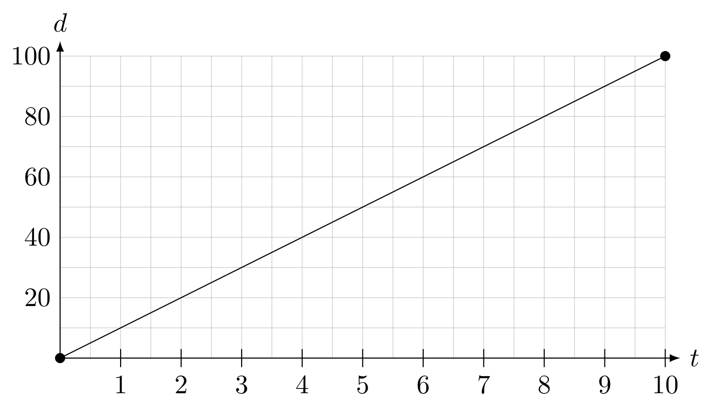


The only things we know about the graph are that the runner started was at the starting line at $t = 0$ and at the finish line (100 meters) at $t = 10$. The speed of 10 meters per second is the result of taking the ratio of distance and time:

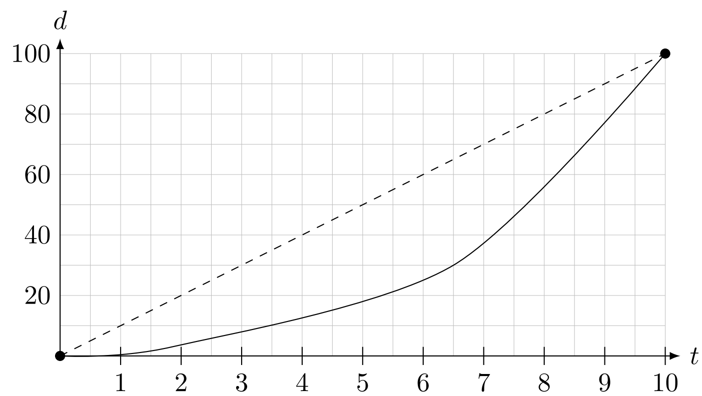
$$\frac{100 \text{ meters}}{10 \text{ seconds}} = 10\,\frac{\text{meters}}{\text{second}}$$

If the original fraction has a familiar feel, that's because it's the formula for slope. It's nothing more than the rise over the run. We can see this more clearly if we connect the two points to create a line.

Instead of $\frac{\Delta y}{\Delta x}$, we have $\frac{\Delta d}{\Delta t}$ because of the variables that we're using. But the directions are still the same.



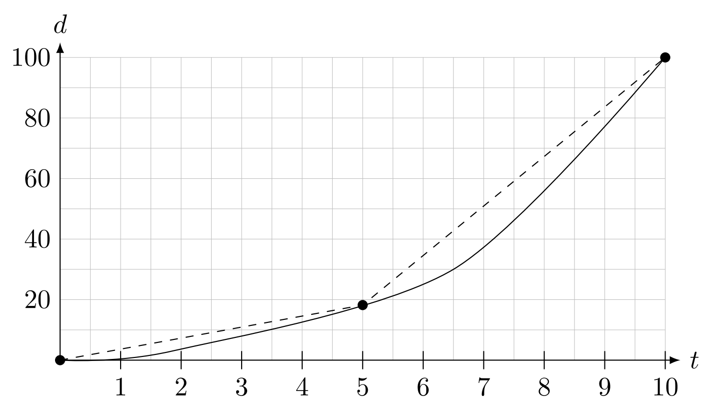
But there's a problem with this graph. It does not represent the actual position of the runner at every moment in time. Sprinters don't run at a constant speed. When they start they're moving at a slow speed, and then speed up as they go. So the real graph may look more like this:



In reality, it looks nothing like this. Sprinters hit their peak speed in the middle of the race and are slowing down at the end. But we're going to go with this because it's visually simpler.

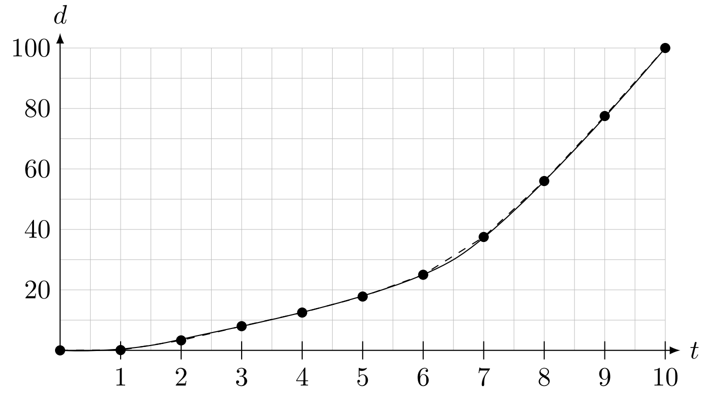
So let's go back and think about what we mean when we say that they ran 10 meters per second. This actually represents the *average* speed over the course of the race. If we only knew the starting point and the ending point, then we can calculate this value as our best estimate of the speed, even though something different may have happened in between.

From this picture, we can try to break things down further. What was the average speed in the first 5 seconds and the last 5 seconds? Now instead of using just the endpoints, we're adding a point in the middle.



Notice that we now have two slopes. One slope for the first part and another slope for the second part. In other words, we now have an average speed for the first half of the race and the second half of the race. The smaller slope at the beginning and larger slope at the end matches with our intuition that the sprinter is moving faster at the end of the race than the beginning.

We can see that theses lines do a better job of capturing the information of the graph, but it's still not very good. We can still see some rather large deviations between the straight lines and the curve. But why stop at just two divisions? Why not make a new division every second?



Now the dashed line matches very closely with the curve. It's still not perfect, but it's getting a lot better. We now have the average speed for every second of the race.

This leads us to an interesting question: What happens if we continue this process? Instead of every second, what if we pushed this to every half second? Or every tenth of a second? Or every hundredth of a second? We can get average speeds for smaller and smaller time intervals. How far can we push this idea? And what is the end result?

If you really want to have some fun with this, look up Zeno's Arrow Paradox.

13.9 Solutions to the "Try It" Examples

1

The selected points may vary, but the slope will always be $\frac{1}{2}$.

From $(-5,-2)$ to $(-3,-1)$: $m = \frac{\Delta y}{\Delta x} = \frac{1}{2}$

From $(-3,-1)$ to $(1,1)$: $m = \frac{\Delta y}{\Delta x} = \frac{2}{4} = \frac{1}{2}$

From $(-1,0)$ to $(5,3)$: $m = \frac{\Delta y}{\Delta x} = \frac{3}{6} = \frac{1}{2}$

2

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{6} = \frac{2}{3}$$

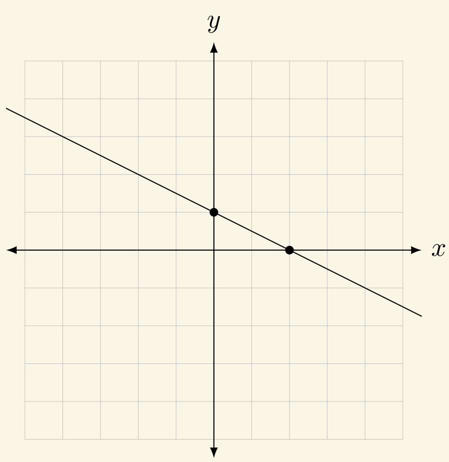
3

$$2x - 5y = 10$$

$$-5y = -2x + 10 \qquad \text{Subtract } 3x \text{ from both sides}$$

$$y = \frac{2}{5}x - 2 \qquad \text{Divide both sides by } -2$$

4



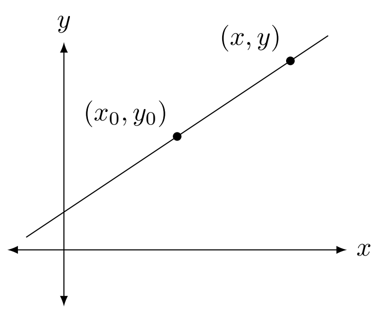
Start Here, Go There: Point-Slope Form

Learning Objectives:

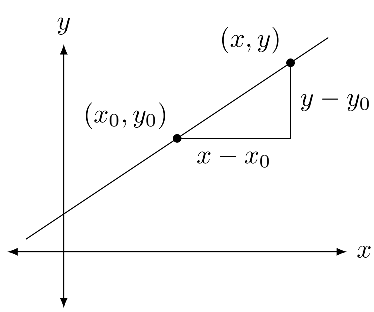
- ☐ Understand point-slope form as an application of the idea of the slope of a line.
- ☐ Determine the point-slope form of a line from given information.
- ☐ Convert equations of lines from point-slope form to slope-intercept form.

The slope-intercept form of a line is useful when using them in practical applications. However, when it comes to writing down the equation of a line, it is often inconvenient. The challenge is that for the slope-intercept form, the location of the initial point is restricted to being the y-intercept, and that's not always true of the available information.

We are going to consider the situation where we know one point on the line and we know what the slope of the line is. The known point will be labeled as (x_0, y_0). We are also going to pick another point on the line and call it (x, y). The goal is to find a mathematical relationship between these two points.



The only piece of information that has not been used is the slope. The slope is the same value no matter which two points on the line are chosen, so we will use these two points. The rise is the change in the y-coordinates, and the run is the change in the x-coordintes. We calculate these by taking the differences between the corresponding coordinates.



If you are not sure about the formula, let (x_0, y_0) be the point $(3, 3)$ and the (x, y) be the point $(6, 5)$, and draw out the picture on a piece of graph paper.

We know that the slope is the change in y divided by the change in x, so we can plug this in and then manipulate the formula.

$$m = \frac{y - y_0}{x - x_0} \implies y - y_0 = m(x - x_0)$$

Definition 14.1. The *point-slope* form of the line that passes through the point (x_0, y_0) with slope m is $y - y_0 = m(x - x_0)$.

1

Once you have the formula, some problems involving the point-slope form of a line simply require to plug in values. For example, the point-slope form of the line that passes through the point $(1, -2)$ with slope $\frac{4}{3}$ is $y + 2 = \frac{4}{3}(x - 1)$.

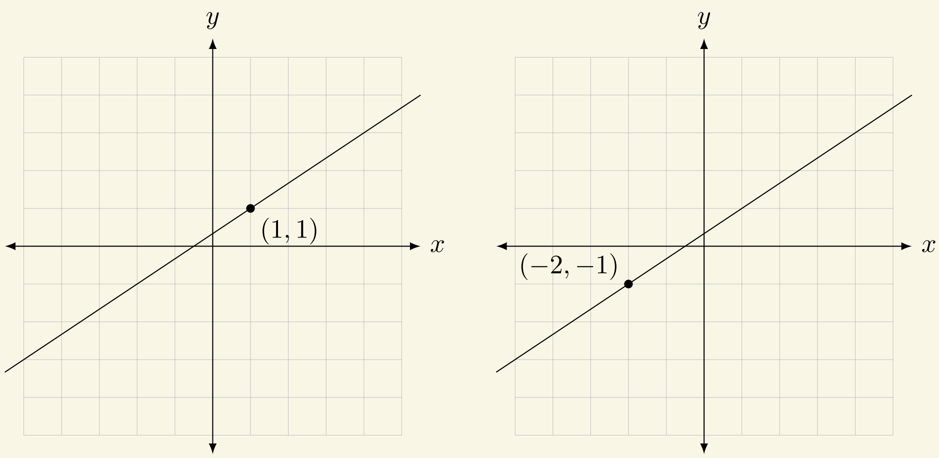
Try it: Determine the point-slope form of the line that passes through the point $(-2, 5)$ with slope -2.

We simplified subtracting a negative to addition:

$$y - (-2) = \frac{4}{3}(x - 1)$$

2

It turns out that a line has multiple representations using the point-slope form, depending on which point is considered to be the initial point. This is much easier to identify from a graph. Notice that the lines in the graphs below are identical.



$y - 1 = \frac{2}{3}(x - 1)$ $\qquad$ $y + 1 = \frac{2}{3}(x + 2)$

Try it: Find at least one more point-slope form for the line above.

Be sure that your initial point is always taken to be on the grid. There are two more possibilities for this example.

3

Because there are multiple representations of a single line, we say that the point-slope form of a line is *not unique*. This means that two different people may end up getting different answers to the same question, and they can both be correct. One way to check that the equations represent the same line is to rewrite the equations in slope-intercept form by solving for y.

$$\begin{aligned} y - 1 &= \tfrac{2}{3}(x - 1) \\ y - 1 &= \tfrac{2}{3}x - \tfrac{2}{3} \\ y &= \tfrac{2}{3}x + \tfrac{1}{3} \end{aligned} \qquad \begin{aligned} y + 1 &= \tfrac{2}{3}(x + 2) \\ y + 1 &= \tfrac{2}{3}x + \tfrac{4}{3} \\ y &= \tfrac{2}{3}x + \tfrac{1}{3} \end{aligned}$$

Try it: Write the equation $y - 3 = \frac{2}{3}(x - 4)$ in slope-intercept form.

14.1 Point-Slope Form - Worksheet 1

1 Determine the point-slope form of the line that passes through the point $(1, 2)$ with slope -2, then convert that equation to slope-intercept form.

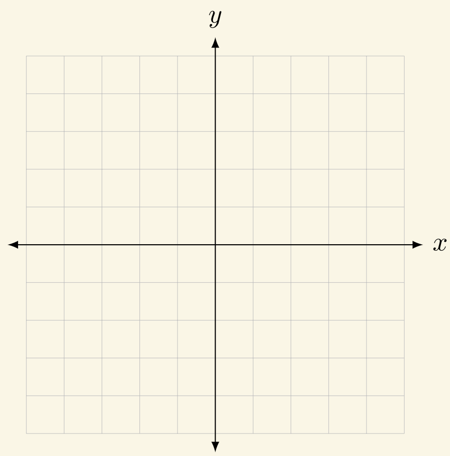
2 Determine the point-slope form of the line that passes through the point $(-4, 1)$ with slope $\frac{5}{2}$, then convert that equation to slope-intercept form.

3 Determine the point-slope form of the line that passes through the point $(2, -3)$ with slope $-\frac{1}{4}$, then convert that equation to slope-intercept form.

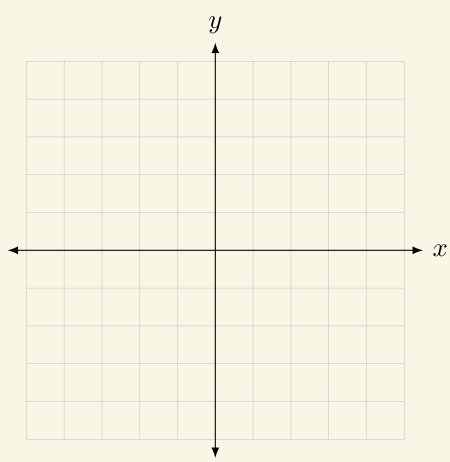
4 Determine the point-slope form of the line that passes through the point $(-3, -2)$ with slope $\frac{5}{3}$, then convert that equation to slope-intercept form.

14.2 Point-Slope Form - Worksheet 2

1 Determine the equation of the line that passes through the point $(-1, 3)$ with slope $-\frac{2}{3}$, then graph it.



2 Identify the point and slope used to create the equation $y - 2 = \frac{3}{2}(x - 4)$, then graph the line.



14.3 Point-Slope Form - Worksheet 3

1 Find two point-slope equations for the line that passes through the points $(-2, 1)$ and $(5, 5)$.

How do you find the slope of a line that passes through two given points?

2 Find two point-slope equations for the line that passes through the points $(-3, 4)$ and $(4, -3)$.

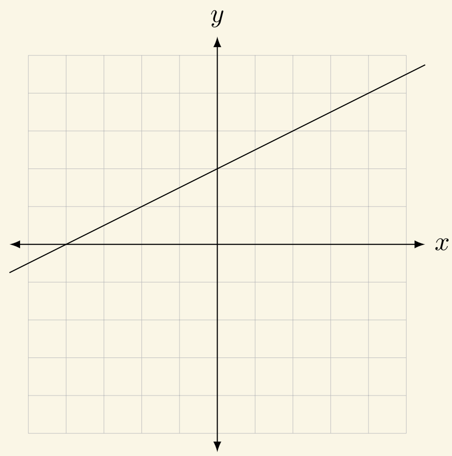
3 Find two point-slope equations for the line that passes through the points $(2, -3)$ and $(-1, 4)$.

4 Find a point-slope equation for the line that passes through the points $(-2, -1)$ and $(3, -1)$.

14.4 Point-Slope Form - Worksheet 4

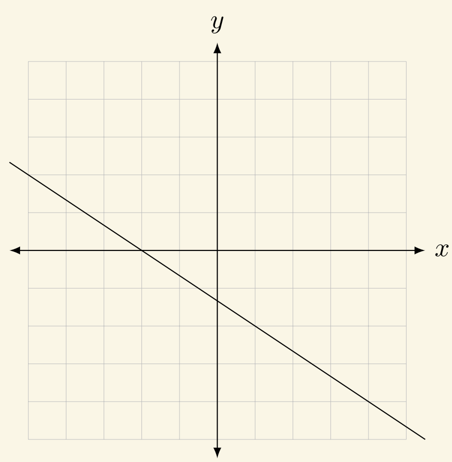
1

Find three different point-slope forms for the given line.



2

Find three different point-slope forms for the given line.



14.5 Point-Slope Form - Worksheet 5

1

Find a point-slope form of the line that passes through the point $(0, b)$ with slope m.

2

Find a point-slope form of the line that passes through the point $(a, 0)$ with slope m.

3

Two lines are parallel if they have the same slope. Find the point-slope form of the line that passes through the point $(2, -1)$ that is parallel to the line $y = 2x - 3$.

4

Two lines are parallel if they have the same slope. Find the point-slope form of the line that passes through the point $(-1, -3)$ that is parallel to the line $y - 2 = \frac{4}{3}(x + 1)$.

14.6 Deliberate Practice: Point-Slope Form

Focus on these skills:

- Present your work legibly.

Instructions: Find a point-slope form of the line that meets the specified conditions, then rewrite it in slope-intercept form.

1. The line that passes through the point $(2, -1)$ with slope $\frac{1}{3}$

2. The line that passes through the point $(-1, 2)$ with slope $\frac{3}{2}$

3. The line that passes through the point $(3, 0)$ with slope -3

4. The line that passes through the points $(2, 4)$ and $(-1, -1)$

5. The line that passes through the points $(1, -2)$ and $(3, 2)$

6. The line that passes through the points $(0, 3)$ and $(2, 0)$

7. The line that passes through the point $(-1, -2)$ that is parallel to $y = -\frac{1}{2}x + 4$

8. The line that passes through the point $(-1, -2)$ that is parallel to $y = \frac{2}{3}x - 1$

9. The line that passes through the point $(2, -3)$ that is parallel to $y + 2 = \frac{5}{2}(x - 2)$

10. The line that passes through the point $(2, -3)$ that is parallel to $y - 1 = -2(x + 3)$

14.7 Closing Ideas

In the last two sections, we have seen two different forms for the equation of a line. Why would we need two different versions of the same thing?

- The slope-intercept form is very useful in practical applications. It has a form that can be interpreted in useful ways.
- The point-slope form is more flexible, and can be applied in both practical and theoretical applications. However, it does not give a unique formula for each line and is often harder to intuitively visualize.

One of the keys to mathematical thinking is having the flexibility to see the same thing in different ways. This is not the only time this theme will come up. Later on, we're going to look at several ways of understanding another concept, and see that each one has their own strengths and weaknesses.

As you continue to learn more mathematics, do not assume that simply because you know how to do things in a certain way that it is the "correct" way of looking at it. It may be that a different approach will yield even better results.

14.8 Solutions to the "Try It" Examples

1

$$y - 5 = -2(x + 2)$$

2

$$y - 3 = \frac{2}{3}(x - 4) \qquad \text{or} \qquad y + 3 = \frac{2}{3}(x + 5)$$

3

$$\begin{aligned} y - 3 &= \frac{2}{3}(x - 4) \\ y - 3 &= \frac{2}{3}x - \frac{8}{3} \\ y &= \frac{2}{3}x + \frac{1}{3} \end{aligned}$$

Replace This With That: The Method of Substitution

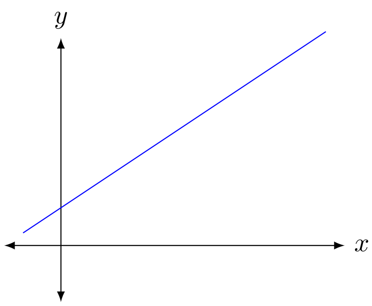
Learning Objectives:

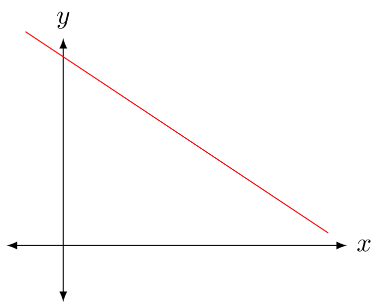
- ☐ Understand the definition of a solution to a simultaneous system of linear equations.
- ☐ Describe the three configurations for a system of two linear equations.
- ☐ Determine the configuration of a system of two linear equations.
- ☐ Solve systems of linear equations using substitution.

In the previous sections, we were looking at lines in isolation of each other. In this section, we're going to look at the ways that two lines can interact with each other.

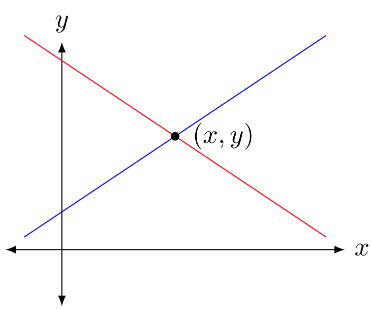
A solution to a linear equation is a pair of values (x, y) that make the equation true. When we talk about a simultaneous system of linear equations, we are looking for a pair of values (x, y) that satisfy all of the equations at the same time. This is probably best understood graphically.

The solution of a single linear equation can be represented as a line. Every point (x, y) on this line can be plugged into that equation and yield a true result. If we add in a second linear equation, we get a different line. These are represented as the two separate graphs below.





When we talk about *simultaneous* solutions, we want to find points that satisfy both equations at the same time. In other words, we're looking for points that are on both lines, and so we can merge the two images into one.



Definition 15.1. A *system of simultaneous linear equations* is a collection of linear equations that are to be solved at the same time, if possible. A *solution* to a system of simultaneous is a point (x, y) that is a solution of all of the equations.

1 We are going to focus on systems of two linear equations with two variables. Consider the following system:

$$\begin{cases} x + y = 2 \\ x - y = 4 \end{cases}$$

We will often use the brackets when working with systems of equations in order to emphasize that they are to be solved simultaneously.

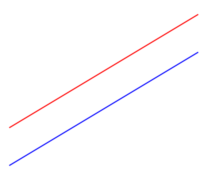
As noted in the definition, a solution must solve all the equations. This means that even though $(1, 1)$ is a solution of $x + y = 2$ and $(2, -2)$ is a solution of $x - y = 4$, neither $(1, 1)$ nor $(2, -2)$ would be a solution to the system because when we plug in the values, we don't solve both.

$$(x, y) = (1, 1) \qquad\qquad (x, y) = (2, -2)$$

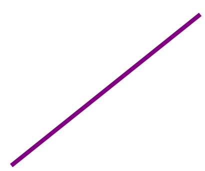
$$\begin{cases} x + y = 2 \\ x - y = 4 \end{cases} \implies \begin{cases} 1 + 1 \overset{\checkmark}{=} 2 \\ 1 - 1 \overset{\times}{=} 4 \end{cases} \qquad \begin{cases} x + y = 2 \\ x - y = 4 \end{cases} \implies \begin{cases} 2 + (-2) \overset{\times}{=} 2 \\ 2 - (-2) \overset{\checkmark}{=} 4 \end{cases}$$

Try it: Show that $(3, -1)$ is a solution to the system of equations by direct substitution.

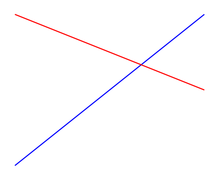
When thinking about different possibilities for solutions, we need to think about the configurations we can get when we graph the two lines together. It turns out that there are three possibilities:



Two different parallel lines



Two overlapping parallel lines



Two non-parallel lines

In the first case, we can see that there are no solutions because there do not exist any points that lie on both lines at the same time. In the second case, we can see that every point on one line is also a point on the other line, so there are infinitely many solutions. In the last case, we can see that there is exactly one point that lies on both lines.

2 If the equations are given in slope-intercept form, it's possible to immediately identify which situation you are in, and it's also possible to use substitution to solve the equations. Suppose that you have the following system of equations, where m_1, m_2, b_1, and b_2 are all constants.

$$\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$$

We can match these up with the images above.

- Two different parallel lines: Same slope, different intercept
- Two overlapping lines: Same slope, same intercept
- Two non-parallel lines: Different slopes

If the lines are not parallel, you can find the solution by substituting one equation into the other.

Try it: Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} y = 4x + 3 \\ y = 2x - 1 \end{cases}$$

Since both expressions is equal to y, they must be equal to each other.

This is an example of solving an equation by substitution. The idea of a substitution here is the same as we used earlier, where we simply replace a symbol in an equation with some other collection of symbols, which then reduces the equation to a single variable equation that we can solve.

3 In general, the equations will not be given to you in slope-intercept form. This means that you will have to choose which variable to solve for and from which equation. There are no rules for this. You can sometimes avoid fractions, but you often can't. Here is the same system of equations solved two ways. Each column represents a different approach, but notice that they end up in the same place.

$$\begin{cases} 3x + 2y = 10 \\ 2x - 3y = -2 \end{cases}$$

Solve for x from the first equation

$$\begin{aligned} 3x + 2y &= 10 \\ 3x &= -2y + 10 \\ x &= -\frac{2}{3}x + \frac{10}{3} \end{aligned}$$

Substitute into the second equation

$$\begin{aligned} 2x - 3y &= -2 \\ 2\left(-\frac{2}{3}y + \frac{10}{3}\right) - 3y &= -2 \\ -\frac{4}{3}y + \frac{20}{3} - 3y &= -2 \\ -\frac{13}{3}y &= -\frac{26}{3} \\ y &= 2 \end{aligned}$$

Substitute back

$$\begin{aligned} x &= -\frac{2}{3}y + \frac{10}{3} \\ &= -\frac{2}{3}(2) + \frac{10}{3} \\ &= -\frac{4}{3} + \frac{10}{3} \\ &= 2 \end{aligned}$$

Solve for y from the first equation

$$\begin{aligned} 3x + 2y &= 10 \\ 2y &= -3x + 10 \\ y &= -\frac{3}{2}x + 5 \end{aligned}$$

Substitute into the second equation

$$\begin{aligned} 2x - 3y &= -2 \\ 2x - 3\left(-\frac{3}{2}x + 5\right) &= -2 \\ 2x + \frac{9}{2}x - 15 &= -2 \\ \frac{13}{2}x &= 13 \\ x &= 2 \end{aligned}$$

Substitute back

$$\begin{aligned} y &= -\frac{3}{2}x + 5 \\ &= -\frac{3}{2}(2) + 5 \\ &= -3 + 5 \\ &= 2 \end{aligned}$$

The solution is $(x, y) = (2, 2)$.

Try it: Solve the system of equations above two more times by solving for x and y from the second equation.

Uncomfortable with fractions? There's a brief fraction review at the end of the section.

Since the parts of the solution are spread out in different areas of the work, it's important to write a conclusion statement to communicate your result.

No matter how you solve the equation, you should get the same final result.

If the system of equations involves two lines with the same slope, but the equations are not written in slope-intercept form, you may not immediately recognize that they have the same slope. However, in the process of working through the algebra, you will end up with all of the variable terms canceling out. If that happens, the equation that you're left with will tell you whether the lines overlap or not. If you get a valid mathematical equation, such as $0 = 0$, then the two lines overlap. If you end up with an invalid mathematical equation, such as $2 = 5$, then the lines do not overlap.

Brief Fraction Review:

We will take a deeper dive into fractions in a later section. For now, we're just going to review the basic mechanics of fractions by looking at some examples.

For addition and subtraction of fractions, you need to use a common denominator. When rewriting a fraction with a different denominator, you must multiply both the numerator and denominator by the same value.

$$\begin{aligned}\frac{2}{3}+\frac{5}{4} &= \frac{2\cdot 4}{3\cdot 4}+\frac{5\cdot 3}{4\cdot 3}\\ &= \frac{8}{12}+\frac{15}{12}\\ &= \frac{23}{12}\end{aligned}$$

$$\begin{aligned}\frac{11}{6}-\frac{1}{2} &= \frac{11}{6}-\frac{1\cdot 3}{2\cdot 3}\\ &= \frac{11}{6}-\frac{3}{6}\\ &= \frac{8}{6}\\ &= \frac{4}{3}\end{aligned}$$

For multiplication of fractions, you need to multiply straight across. If you are multiplying by an integer, you can put that integer over a denominator of 1. If you remember how to reduce before multiplying, that's fine. If you don't, you can just reduce after multiplying. The problems in this section will not involve large enough numbers for that to be important. Make sure to reduce your fractions when you finish.

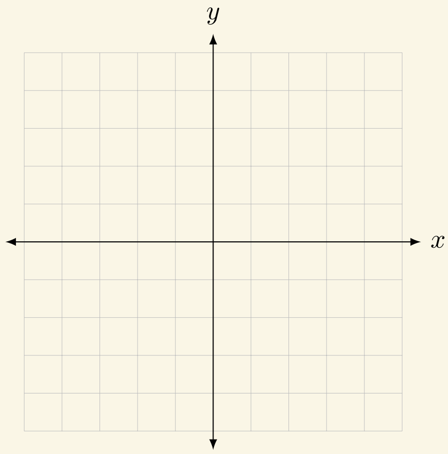
$$\begin{aligned}\frac{3}{4}\cdot\frac{6}{5} &= \frac{3\cdot 6}{4\cdot 5}\\ &= \frac{18}{20}\\ &= \frac{9}{10}\end{aligned}$$

$$\begin{aligned}\frac{7}{2}\cdot 4 &= \frac{7}{2}\cdot\frac{4}{1}\\ &= \frac{7\cdot 4}{2\cdot 1}\\ &= \frac{28}{2}\\ &= 14\end{aligned}$$

15.1 The Method of Substitution - Worksheet 1

1 Graph the lines and describe their configuration.

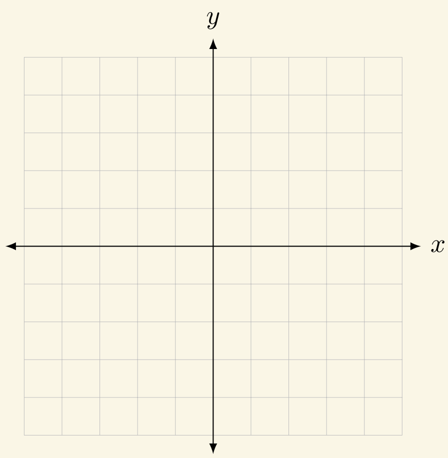
$$\begin{cases} y = \frac{3}{2}x + 1 \\ y = x + 2 \end{cases}$$



The configuration is whether the lines are parallel, overlapping, or intersecting at a single point.

2 Graph the lines and describe their configuration.

$$\begin{cases} y = \frac{1}{3}x + 1 \\ y = \frac{1}{3}x - 3 \end{cases}$$



3 Determine whether $(-2, -1)$ is a solution of the system of equations.

$$\begin{cases} 2x + y = -5 \\ x - 3y = -1 \end{cases}$$

15.2 The Method of Substitution - Worksheet 2

Think about how you might present your work in an organized manner. It's okay to use some words to describe different parts of the calculation.

1 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} y = 3x + 3 \\ y = 2x - 1 \end{cases}$$

2 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} y = -2x + 5 \\ y = x - 1 \end{cases}$$

3 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} y = 3x + 1 \\ y = -x - 2 \end{cases}$$

15.3 The Method of Substitution - Worksheet 3

1

Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$$

2

Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} 3x - 2y = 4 \\ -6x + 4y = -8 \end{cases}$$

3

Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} 2x + y = 3 \\ 4x + 2y = -3 \end{cases}$$

15.4 The Method of Substitution - Worksheet 4

Sometimes, you can plan ahead and make your substitutions less difficult by avoiding fractions.

1 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} 2x - y = 7 \\ -x + 3y = -1 \end{cases}$$

2 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} x + 3y = 0 \\ 2x + 5y = -1 \end{cases}$$

3 Describe the configuration of the following system of equations. If they intersect at a single point, determine the coordinates of that point.

$$\begin{cases} 3x - 4y = -6 \\ -4x + 2y = -5 \end{cases}$$

15.5 The Method of Substitution - Worksheet 5

1

Suppose that $m_1 \neq m_2$. Then the following system of equations intersect at a single point. Determine the coordinates of that point.

b_1 and b_2 are constants.

$$\begin{cases} y = m_1 x + b_1 \\ y = m_2 x + b_2 \end{cases}$$

2

Suppose that the following system of equations intersect at a single point. Determine the coordinates of that point.

To guarantee they intersect at a single point, you need $ad - bc \neq 0$. You should find that expression or its negative appear somewhere in your calculation.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

15.6 Deliberate Practice: Solving by Substitution

Focus on these skills:

- Write the original system of equations.
- Use sentences to organize the steps of your work, including a concluding statement.
- Present your work legibly.

Instructions: Determine the configuration of the system of equations. If they intersect at a single point, determine the coordinates of that point using the method of substitution.

1
$$\begin{cases} x + 2y = 7 \\ -2x + y = 1 \end{cases}$$

2
$$\begin{cases} x - 2y = 4 \\ -3x + 6y = -12 \end{cases}$$

3
$$\begin{cases} 3x - 2y = -8 \\ -x + 3y = 5 \end{cases}$$

4
$$\begin{cases} -2x + 3y = -8 \\ -x + 2y = -5 \end{cases}$$

5
$$\begin{cases} 2x - 3y = 1 \\ 4x - 6y = 4 \end{cases}$$

6
$$\begin{cases} 3x + 4y = 5 \\ 2x + 2y = 3 \end{cases}$$

7
$$\begin{cases} 2x - y = -4 \\ x - 2y = 5 \end{cases}$$

8
$$\begin{cases} -3x - 5y = -3 \\ x + y = 6 \end{cases}$$

9
$$\begin{cases} 3x + 3y = 4 \\ -3x - 2y = -1 \end{cases}$$

10
$$\begin{cases} 4x + 2y = -1 \\ 3x + 2y = 2 \end{cases}$$

15.7 Closing Ideas

This section is primarily an exercise in your core algebra skills. Solve a linear equation for a variable, make a substitution, and then solve for a variable. These are all things that we've already done previously. What's new about it is that those skills are being put together in a brand new way.

It is a common theme in mathematics that we build new ideas on the backs of old ideas. This is also where we can start to identify weaknesses in our understanding, as there may be one step in particular that appears to be more difficult than the others. For example, many students do fine with these problems when all of the numbers are integers, but as soon as fractions start to appear they start to struggle.

This is one of the challenges of the way the field of mathematics is built out. A lot of ideas are built on the foundation of your basic skills. And so even as you move forward into higher levels of math courses, you need to make sure you take the time to go backward to fill in the gaps that you have. And that's one of the main goals of this book.

Everyone has gaps in their knowledge. So if you struggle with something from the past, you can be certain that you're not alone.

As you continue to fill in those gaps, you will start to discover that you're able to make connections between ideas more easily, that you're able to use the tools that you have already developed more efficiently, and that you're able to see how to apply your ideas more effectively. In the end, those are the markers of mature mathematical thinking.

15.8 Solutions to the "Try It" Examples

1

$$(x, y) = (3, -1)$$

$$\begin{cases} x + y = 2 \\ x - y = 4 \end{cases} \implies \begin{cases} 3 + (-1) \stackrel{\checkmark}{=} 2 \\ 3 - (-1) \stackrel{\checkmark}{=} 4 \end{cases}$$

2

$$\begin{cases} y = 4x + 3 \\ y = 2x - 1 \end{cases}$$

Set the equations equal to each other

$$\begin{aligned} 4x + 3 &= 2x - 1 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

Substitute into the first equation

$$\begin{aligned} y &= 4x + 3 \\ &= 4(-2) + 3 \\ &= -5 \end{aligned}$$

The solution is $(x, y) = (-2, -5)$.

3

$$\begin{cases} 3x + 2y = 10 \\ 2x - 3y = -2 \end{cases}$$

Solve for x from the first equation

$$\begin{aligned} 2x - 3y &= -2 \\ 2x &= 3y - 2 \\ x &= \frac{3}{2}y - 1 \end{aligned}$$

Substitute into the second equation

$$\begin{aligned} 3x + 2y &= 10 \\ 3\left(\frac{3}{2}y - 1\right) + 2y &= 10 \\ \frac{9}{2}y - 3 + 2y &= 10 \\ \frac{13}{2}x &= 13 \\ x &= 2 \end{aligned}$$

Substitute back

$$\begin{aligned} x &= \frac{3}{2}y - 1 \\ &= \frac{3}{2}(2) - 1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Solve for y from the second equation

$$\begin{aligned} 2x - 3y &= -2 \\ -3y &= -2x - 2 \\ y &= \frac{2}{3}x + \frac{2}{3} \end{aligned}$$

Substitute into the second equation

$$\begin{aligned} 3x + 2y &= 10 \\ 3x + 2\left(\frac{2}{3}x + \frac{2}{3}\right) &= 10 \\ 3x + \frac{4}{3}x + \frac{4}{3} &= 10 \\ \frac{13}{3}x &= \frac{26}{3} \\ x &= 2 \end{aligned}$$

Substitute back

$$\begin{aligned} y &= \frac{2}{3}x + \frac{2}{3} \\ &= \frac{2}{3}(2) + \frac{2}{3} \\ &= \frac{4}{3} + \frac{2}{3} \\ &= 2 \end{aligned}$$

The solution is $(x, y) = (2, 2)$.

Make it Go Away: The Method of Elimination

Learning Objectives:

☐ Solve systems of linear equations using elimination.

We have seen that we can solve systems of linear equations using substitution, but that one of the challenges of this method was the computational problem of manipulating fractions. The method of elimination is another approach to solving systems of equations that mostly sidesteps the issue (at least at this level), which often makes it a more efficient approach.

The core concept of this method is a slightly different use of axioms of equality that were introduced in Section 1 mixed in with a substitution. We can add the same value to both sides of an equation without losing the equality. The only twist is that we use two different representations in the addition step. Here is how it looks:

If $a = b$ and $c = d$, then $a + c = b + d$.

The challenge is to find useful equations so that adding the equations together gives a helpful result. Here are two examples to compare:

$$\begin{array}{rcrcl} 2x & + & 3y & = & 3 \\ 3x & + & 2y & = & 5 \\ \hline 5x & + & 5y & = & 8 \end{array} \qquad\qquad \begin{array}{rcrcl} 2x & + & y & = & 2 \\ x & - & y & = & 4 \\ \hline 3x & & & = & 6 \end{array}$$

First, notice that the addition in columns is simply an organizational tool. By putting like terms together, it simplifies the process of combining them and avoids errors. Second, notice that the final equation on the right only has one variable even though the two initial equations had two variables. The goal of the method of elimination is to eliminate one of the variables. This allows you to solve for the remaining variable.

1

Here is the worked solution to the system of equations on the right (see above):

$$\begin{array}{rcrcl} 2x & + & y & = & 2 \\ x & - & y & = & 4 \\ \hline 3x & & & = & 6 \\ x & & & = & 2 \end{array} \qquad\qquad \begin{aligned} x - y &= 4 \\ 2 - y &= 4 \quad \text{Substitute } x = 2 \\ -y &= 2 \\ y &= -2 \end{aligned}$$

The solution is $(x, y) = (2, -2)$.

Try it: Solve the system of equations below using the method of elimination.

$$\begin{cases} 3x + y = 3 \\ -3x + 3y = -7 \end{cases}$$

2 Not all systems of equations are in a form that immediately lends itself to the method of elimination. Let's take another look at the example on the left from earlier:

$$\begin{cases} 2x + 3y = 3 \\ 3x + 2y = 5 \end{cases}$$

If we add these equations together directly, nothing will cancel out. But we can multiply either equation by any non-zero number to change its form. And if this is done correctly, it will lead to cancellation.

$$\begin{array}{rcl} 2x + 3y = 3 & \xrightarrow{\times(-3)} & -6x - 9y = -9 \\ 3x + 2y = 5 & \xrightarrow{\times 2} & 6x + 4y = 10 \\ \hline & & -5y = 1 \\ & & y = -\frac{1}{5} \end{array}$$

From here, we can plug in $y = -\frac{1}{5}$ into either of the original equations, but that will lead to fraction manipulations. If we wanted to avoid those, we can just use the same idea to eliminate the y terms instead.

$$\begin{array}{rcl} 2x + 3y = 3 & \xrightarrow{\times(-2)} & -4x - 6y = -6 \\ 3x + 2y = 5 & \xrightarrow{\times 3} & 9x + 6y = 15 \\ \hline & & 5x = 9 \\ & & x = \frac{9}{5} \end{array}$$

This leads to the solution $(x, y) = (\frac{9}{5}, -\frac{1}{5})$.

Try it: Solve the system of equations below using the method of elimination.

$$\begin{cases} 2x + 3y = -7 \\ -3x + y = 3 \end{cases}$$

There's no real "trick" to figuring out the numbers that you need to get the terms to cancel, but there is a general heuristic. Once you've identified the variable you want to eliminate, find the least common multiple of the coefficients, and multiply the equations by the appropriate value to make match, and then make sure that the signs are opposite each other. While some instructors suggest that you can "subtract" the equations, this ends up causing a lot of computational errors, and so it's usually better to use addition.

There's nothing algebraically wrong with subtracting one equation from another. The problem is that it leads to mistakes in the execution that can otherwise be avoided.

16.1 The Method of Elimination - Worksheet 1

1

Solve the system of equations using the method of elimination.

$$\begin{cases} x - y = 5 \\ x + y = 1 \end{cases}$$

2

Solve the system of equations using the method of elimination.

$$\begin{cases} 2x - 2y = 8 \\ -2x + 3y = -6 \end{cases}$$

16.2 The Method of Elimination - Worksheet 2

1 Solve the system of equations using the method of elimination.

$$\begin{cases} 3x + 2y = -5 \\ 2x + y = 2 \end{cases}$$

2 Solve the system of equations using the method of elimination.

$$\begin{cases} 4x + 5y = 0 \\ 2x - 3y = 3 \end{cases}$$

16.3 The Method of Elimination - Worksheet 3

1 Solve the system of equations using the method of elimination.

$$\begin{cases} -x + 2y = -6 \\ 3x - 4y = 1 \end{cases}$$

2 Solve the system of equations using the method of elimination.

$$\begin{cases} -3x + 5y = 3 \\ -2x + 3y = -2 \end{cases}$$

16.4 The Method of Elimination - Worksheet 4

1

Attempt to solve the system of equations using the method of elimination. Explain what happened and the conclusion that you can draw from it.

$$\begin{cases} 2x + 4y = 8 \\ -x - 2y = -4 \end{cases}$$

2

Attempt to solve the system of equations using the method of elimination. Explain what happened and the conclusion that you can draw from it.

$$\begin{cases} x - 3y = 2 \\ -3x + 9y = 5 \end{cases}$$

16.5 The Method of Elimination - Worksheet 5

1

Solve the system of equations using the method of elimination.

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

2

Use the formula you derived from the previous problem to solve the system of equations. Do you find it easier to use the formula or to use the method of elimination?

$$\begin{cases} -2x + 3y = 5 \\ 3x - 5y = -6 \end{cases}$$

16.6 Deliberate Practice: Solving by Elimination

Focus on these skills:

- Write the original system of equations.
- Indicate the manipulations of the equations in the process of eliminating the variables.
- Present your work legibly.

Instructions: Solve the system of equations by elimination. If the system has either zero or infinitely many solutions, explain how you can determine this information from your calculations.

1
$$\begin{cases} x + 2y = 7 \\ -2x + y = 1 \end{cases}$$

2
$$\begin{cases} x - 2y = 4 \\ -3x + 6y = -12 \end{cases}$$

3
$$\begin{cases} 3x - 2y = -8 \\ -x + 3y = 5 \end{cases}$$

4
$$\begin{cases} -2x + 3y = -8 \\ -x + 2y = -5 \end{cases}$$

5
$$\begin{cases} 2x - 3y = 1 \\ 4x - 6y = 4 \end{cases}$$

6
$$\begin{cases} 3x + 4y = 5 \\ 2x + 2y = 3 \end{cases}$$

7
$$\begin{cases} 2x - y = -4 \\ x - 2y = 5 \end{cases}$$

8
$$\begin{cases} -3x - 5y = -3 \\ x + y = 6 \end{cases}$$

9
$$\begin{cases} 3x + 3y = 4 \\ -3x - 2y = -1 \end{cases}$$

10
$$\begin{cases} 4x + 2y = -1 \\ 3x + 2y = 2 \end{cases}$$

16.7 Closing Ideas

This section created a new toolbox of ideas to solve the types of equations that you already knew how to solve. In that sense, this section is extraneous. Why learn a new way to do something you already know how to do? But in another sense, this is a critical section because it teaches you that sometimes the way you've done things in the past isn't necessarily the way you want to do things in the future.

The method of substitution works for a wide range of problems, and it follows a very straightforward pattern: Solve for one of the variables and plug it into the other equation. The method of elimination requires some intuition and intellectual flexibility. You have to actively assess the situation and determine what choice of multiplication values will cause the elimination to happen. But if you choose wisely, the calculations are much simpler. In fact, with a bit of practice you can solve some of these problems mentally (as long as the numbers aren't too big). You would find that to be extremely difficult if you were thinking about using the method of substitution.

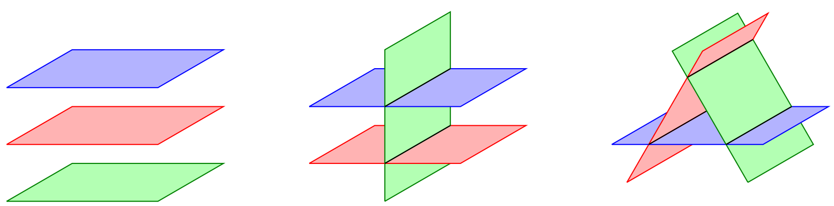
So consider this section to be one about simply expanding your toolbox of techniques. Instead of being confined to just one way of looking at these problems, you now have two. You have the option of looking at the equations to decide whether substitution or elimination makes more sense, and you have the freedom to pick the method that suits you the best. As you get further in your mathematical studies, you will find more and more situations where there are multiple approaches to solve a problem, and that you'll be making more active decisions about the methods and techniques that you apply.

16.8 Going Deeper: Three Dimensional Systems

In the last couple sections, we've looked at systems of two linear equations with two variables. We saw that there was a geometric aspect to understanding these systems by looking at the graphs of the lines, and we also saw that there was an algebraic aspect by looking at the methods of solving these systems. We are going to take a look at what happens when we increase the complexity by looking at systems of three equations and three variables. Here is an example:

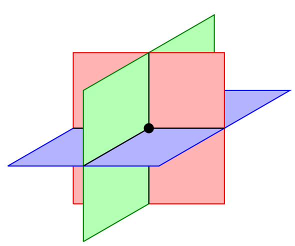
$$\begin{cases} 2x - y - z = 1 \\ x + 2y + z = 4 \\ x + y - z = 4 \end{cases}$$

We will first wrap our minds around the geometric understanding of these equations. It turns out that these formulas form planes and that planes are the natural shape to generalize the "linearness" of lines in two variables. When we think about three simultaneous systems of equations in three variables, we're looking for a point that lies on all three planes. Since the number of dimensions has increased, it makes sense that the number of possible arrangements would also increase. However, we still have the three same basic classifications that we had before. There will either be zero solutions, one solution, or infinitely many solutions.



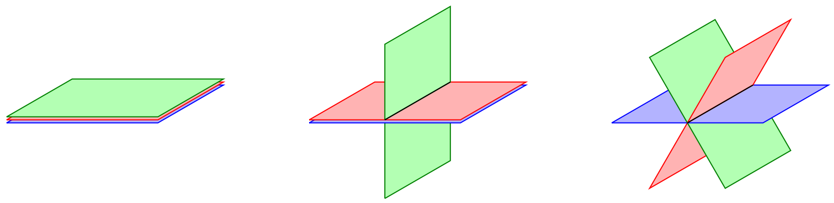
In each of the systems above, there are no solutions. This may seem unusual at first, since there are clearly some points of intersection. However, we are working with *simultaneous* systems of equations, which means that we're looking for points that lie on *all three* planes, not just pairs of planes. Notice that we cannot use the simple heuristic of just thinking about whether the planes are parallel, because the diagram on the right shows that it's possible to have no solutions even though none of the planes are parallel to each other.

A key observation about these diagrams is that when there are intersections between planes, it forms a line. This is always true of the intersection of two planes, which is an idea we'll come back to when we look at the algebra.



This is an example of a system with one solution. Although everything is draw to be at right angles with each other, there is a lot of flexibility in terms of the relative positions of the planes.

You should be able to visualize each pair of planes forming a line, and that each of those lines cross each other in exactly one place.



The diagram stacks are meant to indicate that the planes overlap each other. It's hard to visually represent that without having at least some separation between them.

Lastly, we have these configurations to get infinitely many points of intersection. The diagram on the left has all three planes overlapping each other. The middle diagram has two planes overlapping each other with the third plane cutting across them. The last diagram has all three planes intersecting along the same line. Even though these all have infinitely many solutions, there is a distinction between the first one and the last two. Intuitively, what we're seeing are different numbers of dimensions in the overlap. The intersection of the first diagram is a two-dimensional plane, whereas the intersection of the last two are a one-dimensional line.

This suggests that we might call the single point of intersection a zero-dimensional object.

When it comes to solving these systems algebraically, it's just the methodical application of either substitution or elimination to reduce the number of variables. Conceptually, we're going to find the intersection of planes to get lines, then find the intersection of lines to get points. When working through the process, it's much more about keeping the work organized so that you don't get lost.

Solving Three Dimensional Systems by Substitution

$$\begin{cases} 2x - y - z = 1 \\ x + 2y + z = 4 \\ x + y - z = 4 \end{cases}$$

We will solve the first equation for z:

Just as with the systems with two equations, there are many paths to the solution. You are encouraged to try other paths for practice.

$$\begin{aligned} 2x - y - z &= 1 \\ z &= 2x - y - 1 \end{aligned}$$

Then we will substitute this into the other equations and simplify:

$$\begin{aligned} x + 2y + z &= 4 & x + y - z &= 4 \\ x + 2y + (2x - y - 1) &= 4 & x + y - (2x - y - 1) &= 4 \\ 3x + y &= 5 & -x + 2y &= 3 \end{aligned}$$

We can now solve for y in the first of the new equations:

$$\begin{aligned} 3x + y &= 5 \\ y &= -3x + 5 \end{aligned}$$

And plug that into the second of the new equations and solve for x:

$$\begin{aligned} -x + 2y &= 3 \\ -x + 2(-3x+5) &= 3 \\ -7x + 10 &= 3 \\ x &= 1 \end{aligned}$$

We can now work backwards to get the remaining variables.

This process is often called *back substitution.*

$$\begin{aligned} y &= -3x + 5 \\ &= -3(1) + 5 \\ &= 2 \end{aligned} \qquad \begin{aligned} z &= 2x - y - 1 \\ &= 2(1) - (2) - 1 \\ &= -1 \end{aligned}$$

The solution of the system of equations is $(x, y, z) = (1, 2, -1)$.

Solving Three Dimensional Systems by Elimination

$$\begin{cases} 2x - y - z = 1 \\ x + 2y + z = 4 \\ x + y - z = 4 \end{cases}$$

We will first eliminate z twice by first combining the first two equations then the last two equations:

$$\begin{array}{r} 2x - y - z = 1 \\ x + 2y + z = 4 \\ \hline 3x + y \phantom{{}+z} = 5 \end{array} \qquad \begin{array}{r} x + 2y + z = 4 \\ x + y - z = 4 \\ \hline 2x + 3y \phantom{{}+z} = 8 \end{array}$$

We will now combine these equations to eliminate x:

$$\begin{array}{r} 3x + y = 5 \\ 2x + 3y = 8 \\ \hline \end{array} \quad \begin{array}{c} \xrightarrow{\times 2} \\ \xrightarrow{\times(-3)} \end{array} \quad \begin{array}{r} 6x + 2y = 10 \\ -6x - 9y = -24 \\ \hline -7y = -14 \\ y = 2 \end{array}$$

Then combine them again to eliminate x:

$$\begin{array}{r} 3x + y = 5 \\ 2x + 3y = 8 \\ \hline \end{array} \quad \begin{array}{c} \xrightarrow{\times(-3)} \\ \xrightarrow{\times 1} \end{array} \quad \begin{array}{r} -9x - 3y = -15 \\ 2x + 3y = 8 \\ \hline -7x \phantom{{}+3y} = -7 \\ x = 1 \end{array}$$

While we could be stubborn and go through this process again to solve for z using elimination, it makes much more sense to simply substitute the known values into one of the original equations

You should be able to plug the values of x and y into any of the original equations and get the same result.

to solve for z. We will use the first equation.

$$\begin{aligned} 2x - y - z &= 1 \\ 2(1) - (2) - z &= 1 \\ -z &= 1 \\ z &= -1 \end{aligned}$$

The solution of the system of equations is $(x, y, z) = (1, 2, -1)$.

16.9 Solutions to the "Try It" Examples

1

$$\begin{array}{rcrcr} 3x & + & y & = & 3 \\ -3x & + & 3y & = & -7 \\ \hline & & 4y & = & -4 \\ & & y & = & -1 \end{array}$$

$$\begin{aligned} 3x + y &= 3 \\ 3x + (-1) &= 3 \quad \text{Substitute } x = 2 \\ 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

The solution is $(x, y) = (\frac{4}{3}, -1)$.

2

$$\begin{cases} 2x + 3y = -7 \\ -3x + y = 3 \end{cases}$$

$$\begin{array}{rcl} 2x + 3y = -7 & \xrightarrow{\times 3} & 6x + 9y = -21 \\ -3x + y = 3 & \xrightarrow{\times 2} & -6x + 2y = 6 \\ \hline & & 11y = -15 \\ & & y = -\frac{15}{11} \end{array}$$

$$\begin{array}{rcl} 2x + 3y = -7 & \xrightarrow{\times 1} & 2x + 3y = -7 \\ -3x + y = 3 & \xrightarrow{\times (-3)} & 9x - 3y = -9 \\ \hline & & 11x = -16 \\ & & x = -\frac{16}{11} \end{array}$$

The solution is $(x, y) = (-\frac{16}{11}, -\frac{15}{11})$.

Facing Your Fear: Fraction Basics

Learning Objectives:

- ☐ Understand and represent fractions and mixed numbers using parts of a whole.
- ☐ Represent fractions and mixed numbers on a number line.
- ☐ Reduce fractions, including fractions with variables.

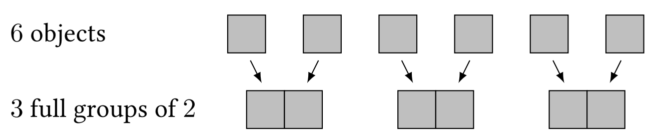
Most students don't like fractions. This is just a reality. This is unfortunate because fractions are everywhere in mathematical applications. A lot of students really only know fractions as a system of rules for manipulations, and have little intuition for what they represent and how we use them. Over the next few sections, we're going to take a closer look at fractions with the goal of making fractions more understandable.

If you are looking for long lists of practice problems to drill your fraction skills, try looking on the internet.

Definition 17.1. A fraction is a mathematical expression of the form $\frac{a}{b}$ where $b \neq 0$. We call a the numerator of the fraction and b the denominator of the fraction.

We do not put conditions on what types of mathematical objects a and b could be. While they are usually integers, they can be any numerical quantity or variable expression.

At its core, fractions are a representation of division. The fraction $\frac{6}{2}$ can be interpreted as asking the question, "How many groups of 2 can we make if we have 6 objects?" The answer is 3, and we can visualize it with a simple picture.

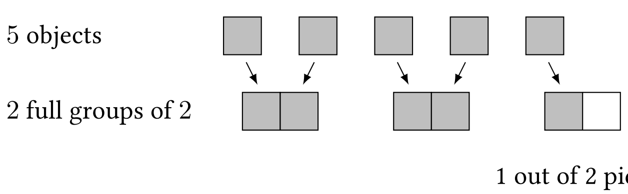


$$\frac{6}{2} = 3$$

When this works out evenly, our answer is an integer. When it doesn't, then we can also use fractions to communicate the leftover pieces. The fraction $\frac{5}{2}$ asks the question "How many groups of 2 can we make if we have 5 objects?" The answer is $2\frac{1}{2}$. The $\frac{1}{2}$ part simply means that we have one out of the two pieces needed to create another group. We can visualize the missing piece with an unshaded box.

Mathematically, we use the following notation for the set of integers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$



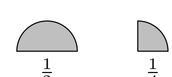
$$\frac{5}{2} = 2\tfrac{1}{2}$$

Numbers like $2\frac{1}{2}$ that are a mixture of an integer part and a fraction are called mixed numbers. They are commonly used in applications (such as stock prices and lumber measurements), but they are algebraically cumbersome to use. In fact, the mixed number $2\frac{1}{2}$ is really $2 + \frac{1}{2}$ when it comes to algebraic manipulations. So we often leave fractions involving integers in their "improper" form.

An "improper fraction" is a fraction where the numerator is larger than the denominator. It's an unfortunate name, because there's nothing improper about it.

The visualization of fractions above is known as "parts of a whole." The idea is that you have some concept of what a "whole" grouping looks like, and you're trying to fill that in with the

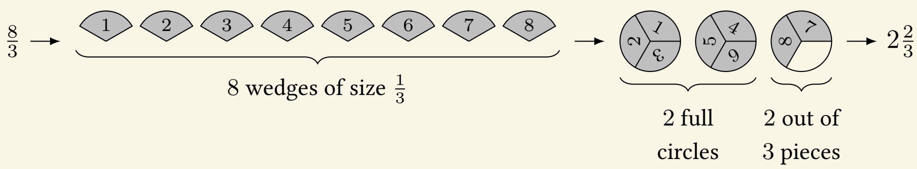
When using boxes, you don't know what a whole collection looks like unless you're explicitly told. When using a circle, you get an automatic visual clue.



number of "parts" that you have. It is often the case that we use circles when talking about parts of a whole because it's more intuitive if the whole is always the same final shape.

Converting between improper fractions and mixed numbers can be done using strict arithmetic methods, but it's also important to develop that base intuition of what fractions are and how they behave. So rather than giving an explanation of converting between improper fractions and mixed numbers in words, we're going to do it in pictures.

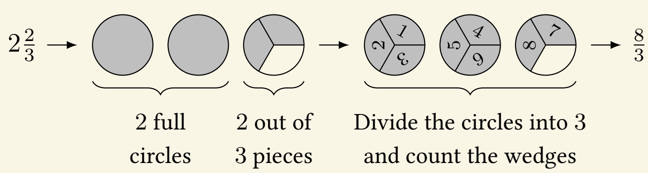
1 Here is the conversion of an improper fraction to a mixed number:



Try it: Convert $\frac{11}{4}$ from an improper fraction to a mixed number using a diagram.

Use the example as your model.

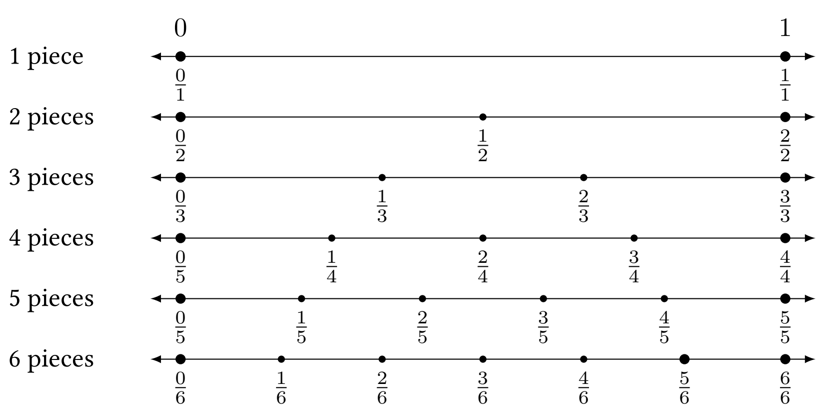
2 Here is the conversion of a mixed number to an improper fraction:



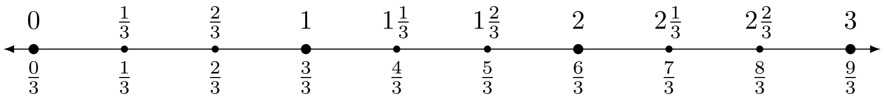
Try it: Convert $2\frac{3}{4}$ from a mixed number to an improper fraction using a diagram.

The visualization of fractions as wedges of circles is one of two primary visualizations that we use for fractions. The second comes from the number line. A number can be represented by a position on the number line. Here are some examples:

Remember that having multiple ways of thinking about the same concept gives you a deeper perspective from which to draw your ideas.

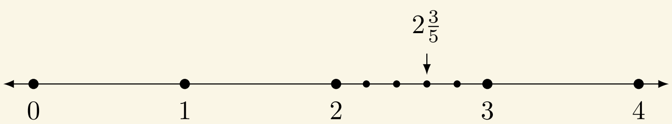


For improper fractions and mixed numbers, we can continue doing this process and extend beyond the interval from 0 to 1.



The integers are marked with a slightly larger dot as a helpful visual marker. When you draw these by hand, it's not necessary to do that, but it can be helpful.

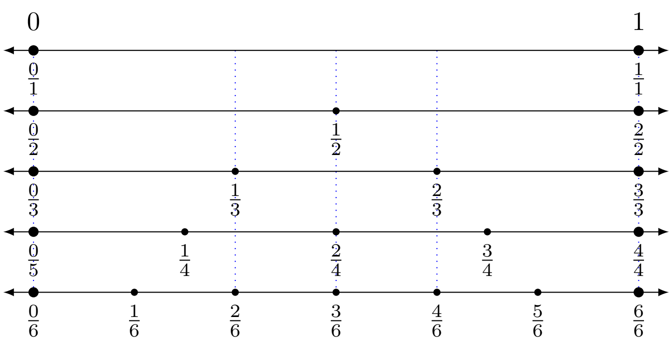
3 When locating numbers on a number line, you do not need to mark all of the subdivisions between the integers, just the subdivisions between the integers you are focused on for your value. Here is how $2\frac{3}{5}$ could be represented:



Try it: Represent $\frac{11}{3}$ on a number line.

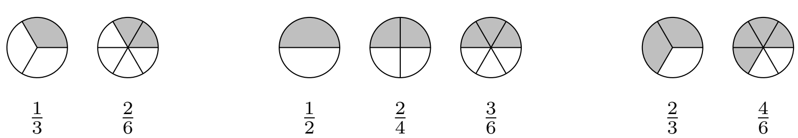
You do not need to show any work for converting your improper fraction to a mixed number in this problem. That is something you should be able to do mentally.

Let's take another look at the number lines. Notice specifically that several of the fractions that are in the same position even though they have different denominators.



If we were to keep going, we would find even more matches.

In fact, we can use the parts of a whole picture to visualize these relationships as well. Here are different fraction representations that result in the same amount shaded in:



What this means is that there are multiple fractions that represent the same number:

$$\frac{1}{3} = \frac{2}{6} = \cdots \qquad \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots \qquad \frac{2}{3} = \frac{4}{6} = \cdots$$

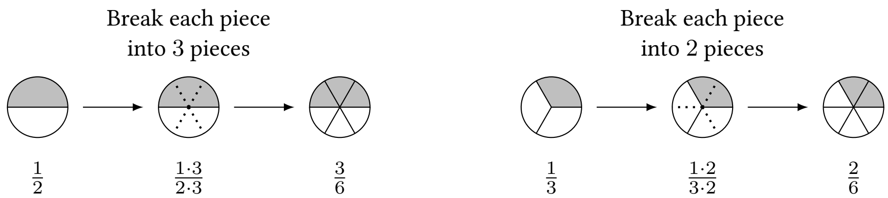
Definition 17.2. Two fractions are *equivalent* if they represent the same quantity.

This leads us to an important question: When are two fractions equivalent? If we look at the pattern of values, we can see that the fractions are related to each other by multiplying the numerator and the denominator by the same value.

$$\frac{1}{3} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6} \qquad \frac{1}{2} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4}$$

$$\frac{1}{2} = \frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \qquad \frac{2}{3} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{4}{6}$$

There are many ways to think about this. One way to think about it is that we're taking the parts of a whole diagram and cutting them into extra pieces without changing the shaded area. Here are two examples of this:



This gives us a general pattern that we can follow to generate equivalent fractions. If x is any non-zero number and $\frac{a}{b}$ is any fraction, then we have

$$\frac{a}{b} = \frac{a \cdot x}{b \cdot x} = \frac{ax}{bx}.$$

When students are introduced to this, it is usually done with non-zero integers. However, this is true regardless of the value of x. In fact, it's even true when x is a non-zero variable or variable expression. In other words, the following are all equivalent:

$$\frac{2}{3} = \frac{4}{6} = \frac{2x}{3x} = \frac{4x}{6x} = \frac{4x^3y^2}{6x^3y^2} = \frac{2(x-3)}{3(x-3)}$$

This idea can also be turned around. Sometimes there are complicated fractions with common factors that can be "cancelled out" to leave you with a simpler expression. But this can only be done if the numerator and denominator have the same factor and that factor is being multiplied by the rest of the numerator and denominator.

We often cancel and reduce in the same step to save time and space. The book will continue to show them separately for clarity.

$$\frac{4}{6} = \frac{2 \cdot \cancel{2}}{3 \cdot \cancel{2}} = \frac{2}{3}$$

$$\frac{4}{6} = \frac{2 \cdot 2}{3 \cdot 2} = \frac{2 \cdot \cancel{2}}{3 \cdot \cancel{2}} = \frac{2}{3} \qquad \frac{4x}{6x} = \frac{2 \cdot 2x}{3 \cdot 2x} = \frac{2 \cdot \cancel{2x}}{3 \cdot \cancel{2x}} = \frac{2}{3}$$

4 The key to reducing fractions is identifying the common factors. This is the same process as when we were factoring polynomial expressions.

$$\frac{24}{60} = \frac{2 \cdot 5}{12 \cdot 5} = \frac{2 \cdot \cancel{5}}{12 \cdot \cancel{5}} = \frac{2}{5} \qquad \frac{8x^2y^3}{14xy^6} = \frac{4x \cdot 2xy^3}{7y^3 \cdot 2xy^3} = \frac{4x \cdot \cancel{2xy^3}}{7y^3 \cdot \cancel{2xy^3}} = \frac{4x}{7y^3}$$

Try it: Completely reduce the fractions $\frac{6}{8}$ and $\frac{10x^4}{25x^2}$.

"Completely reduce" means to reduce until it cannot be reduced any further.

17.1 Fraction Basics - Worksheet 1

Diagrams are not forever. But they are important enough to practice a few times.

1 Convert $\frac{15}{4}$ from an improper fraction to a mixed number using a diagram.

2 Convert $\frac{17}{3}$ from an improper fraction to a mixed number using a diagram.

3 Convert $3\frac{1}{5}$ from a mixed number to an improper fraction using a diagram.

4 Convert $2\frac{3}{7}$ from a mixed number to an improper fraction using a diagram.

5 Represent $2\frac{3}{4}$ on a number line.

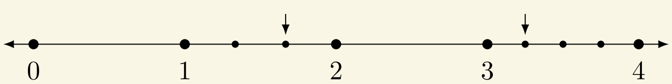
17.2 Fraction Basics - Worksheet 2

1 Convert $\frac{12}{7}$ from an improper fraction to a mixed number using a diagram.

2 Suppose you are given the fraction $\frac{a}{b}$ where a and b are both integers and $b \neq 0$. Describe a calculation that would give you the corresponding mixed number without drawing out a diagram.

Do not divide by zero!

3 Determine the values corresponding to the positions indicated in the diagram below.



4 Completely reduce the fractions $\frac{21}{28}$ and $\frac{18}{48}$.

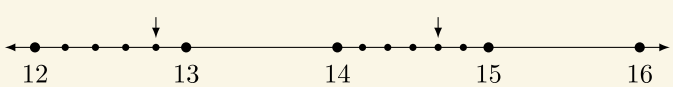
5 Completely reduce the fractions $\frac{8x^2}{6x^4}$ and $\frac{15x^5}{35x^2}$.

17.3 Fraction Basics - Worksheet 3

1 Convert $2\frac{3}{8}$ from a mixed number to an improper fraction using a diagram.

2 Suppose you are given the fraction $a\frac{b}{c}$ where a, b, and c are all integers and $c \neq 0$. Describe a calculation that would give you the corresponding improper fraction without drawing out a diagram.

3 Determine the values corresponding to the positions indicated in the diagram below.



4 Completely reduce the fractions $\frac{25}{40}$ and $\frac{12}{27}$.

5 Completely reduce the fractions $\frac{10x^3y^2}{25xy^5}$ and $\frac{21a^3b^3}{49a^3b^2}$.

17.4 Fraction Basics - Worksheet 4

1 Convert $\frac{23}{5}$ and $\frac{25}{7}$ from improper fractions to mixed numbers without drawing a diagram.

2 Convert $3\frac{2}{9}$ and $4\frac{4}{5}$ from mixed numbers to improper fractions without drawing a diagram.

3 Completely reduce the fractions $\frac{35}{60}$ and $\frac{28m^2n^3}{36mn^8}$.

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\frac{4xy}{12x^2y^3} = \frac{4xy}{3xy^2 \cdot 4xy} = \frac{\cancel{4xy}}{3xy^2 \cdot \cancel{4xy}} = 3xy^2$$

This mistake is fairly common.

5 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\frac{20x^2}{5x^5} = \frac{4 \cdot 5x^2}{x^3 \cdot 5x^2} = \frac{4 \cdot \cancel{5x^2}}{x^3 \cdot \cancel{5x^2}} = 4x^3$$

17.5 Fraction Basics - Worksheet 5

1 Completely reduce the fractions $\frac{4x^2}{6x^4}$ and $\frac{4(x+2)^2}{6(x+2)^4}$.

There is no conceptual difference between these two calculations.

2 Completely reduce the fractions $\frac{12xy^4}{18x^4y^2}$ and $\frac{12(x+2)(x-3)^4}{18(x+2)^4(x-3)^2}$.

Do you understand the pattern?

3 Completely reduce the fractions $\frac{9ab^3}{12a^2b}$ and $\frac{9\sin(x)\cos^3(x)}{12\sin^2(x)\cos(x)}$.

Note: $\sin^n(x)$ is a shorthand for $(\sin(x))^n$ and $\cos^n(x)$ is a shorthand for $(\cos(x))^n$.

4 Completely reduce the fraction $\frac{x^2+5x+6}{x^2-3x-10}$.

Hint: Factor, then trust your experience to guide you from there.

17.6 Deliberate Practice: Reducing Fractions

Focus on these skills:

- Write the original expression.
- Show the cancellation.
- Present your work legibly.

Instructions: Perform the indicated calculation..

1. $$\frac{12x^3y^2}{20x^2y^3}$$

2. $$\frac{24a^5b^3}{42a^3b^3}$$

3. $$\frac{15p^3q^4}{5p^6q^5}$$

4. $$\frac{12x^5y^4z}{18xy^2z^4}$$

5. $$\frac{15a^3b^2}{35ab^2c^4}$$

6. $$\frac{24m^2n^4p^3}{10m^5p^4}$$

7. $$\frac{8(x+3)^2(x-4)^5}{30(x+3)^3(x-4)^2}$$

8. $$\frac{18(x+3)^2(x-4)^5}{45(x+3)^3(x-4)^2}$$

9. $$\frac{30(x+1)^2(x-3)^5}{16(x+1)^3(x-3)^2}$$

10. $$\frac{49x^2(x-1)^4(x+2)^5}{28x^5(x-1)^2(x-1)^5}$$

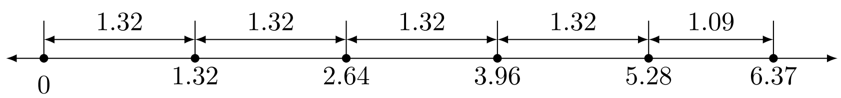
17.7 Closing Ideas

Things get a bit more complicated when we start introducing decimals into fractions. The basic concept of thinking about fractions as division still applies, but it becomes more difficult to create diagrams.

And fractions inside of fractions is even more complicated.

Consider the fraction $\frac{6.37}{1.32}$. This is asking "how many groups of size 1.32 can be made if you have 6.37 items?" Conceptually, it makes sense, and we can come up with some basic analogies to help us think through it, such as "How many items can you buy for \$1.32 if you only have \$6.37 to spend?"

While we really can't draw an effective parts of a whole diagram for this problem, it's possible to set this up with a number line diagram to represent division. And we can push forward with manipulating the symbols the same way we did for fractions.



$$\frac{6.37}{1.32} = 4\frac{1.09}{1.32}$$

We basically never write mixed numbers with decimals. That one is there just to emphasize that the ideas are the same.

In real life, we usually reach for a calculator for fractions involving decimals. We invented the technology precisely to help us with those situations. But this doesn't negate the importance of having a conceptual foundation. While calculators can give us numerical results, it does not have the ability to conceptualize the idea of a fraction.

In this section, we used the intuition of "parts of a whole" with integer values to develop a pattern for manipulating fractions with variables. But we did not use any diagrams to try to represent those fractions. The reason is that our diagrams basically fall apart on us. Here are how different the pictures can look depending on the value of the variable:

The absence of concrete images makes fraction manipulations very abstract when we start involving variables. And this is where algebraic fluency really needs to kick in. As we go through the next few sections, we will be continuing to explore fractions. While we will be primarily working in the context of integer fractions, it is important to keep in mind that the algebraic processes that we're developing can also be applied to variable fractions, and you will have to trust your algebraic fluency more and more as you get further along.

Learning to trust yourself with algebra is a key step, but it can be very difficult if you've spent many years struggling with it. It's kind of like removing training wheels from a bike. Losing confidence and thinking about falling over can lead to falling over. It takes time and practice (and sometimes a certain number of failures) to build up confidence. But it can be done.

17.8 Going Deeper: Reducing Rational Expressions

Usually, when students think of calculus, they think of extremely elaborate calculations and complicated algebraic manipulations. And while there are certainly some aspects of the course for which that is true, it turns out that the core concepts of calculus are ideas that you can understand without needing a lot of algebra.

Derivatives are basically slopes. Integrals are basically areas. The trick to getting there is the concept of a *limit*, which can be understood intuitively without much algebra at all.

One of the largest hurdles students face when they get to calculus is not the calculus, but the algebra. It is very possible to get all the way to calculus without having attained algebraic fluency. As the expressions get more complex, the error rate goes up. And this, much more than the calculus, ends up holding students back.

One area of algebra that students struggle with in particular is the manipulation of fractions. This is an extremely important skill because the main idea of the first half of calculus (differential calculus) is defined as a fraction. In theory, you already have all the skills required to perform the algebraic part of these calculations. You know how to perform a substitution, multiply polynomials, and combine like terms. The remaining algebraic step is to reduce the algebraic fraction.

The derivative of the function f is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

In all of the examples and problems in this section, you were given a fraction where all the numerator and denominator were simply products of algebraic terms. This helps to give students the practice of matching up terms and reducing correctly in these situations. But it also leads students down the path of incorrect manipulations when they do not fully understand the cancellation process.

All examples and problems except for Problem 4 of Worksheet 5.

Over the next several sections, the "Going Deeper" sections are going to focus specifically on dealing with fractions involving polynomials, which are often called *rational expressions.* These sections are aimed at students who are on the pathway towards calculus to help develop specific algebraic skills that are useful along that pathway.

Reducing fractions is a mathematical process built on multiplication. You should be looking for terms that are multiplicative in the numerator and the denominator. More precisely, you need to be able to factor identical terms from the numerator and the denominator. This is why we chose to explicitly write out the products before reducing in this section.

Whether you see it as multiplication or division depends on how you choose to view it. We're just going to go with multiplication.

Most of the errors that come from reducing fractions incorrectly result from different forms of simply crossing off terms that look the same in the numerator and the denominator. When you look at these calculations, you will probably immediately recognize them as being wrong. But it's easy to identify errors when it's someone else's work and you're being told that these are errors. It's often more difficult when you're looking at your own work.

A Collection of Errors

- Canceling addition: Do not cancel out terms when they are being added or subtracted. It must always be a multiplicative term to cancel.

$$\frac{x^2+5x+4}{x^2-4x+4} \overset{\times}{=} \frac{x^2+5x\cancel{+4}}{x^2-4x\cancel{+4}} = \frac{x^2+5x}{x^2-4x}$$

- Partial cancellation: This is another version of the previous error. Even though the terms are being multiplied by something, that something isn't the entire remainder of the numerator and the denominator. The mistake here is canceling out part of an additive term.

$$\frac{3x+1}{2x+1} \overset{\times}{=} \frac{3\not{x}+1}{2\not{x}+1} = \frac{3+1}{2+1} = \frac{4}{3}$$

- Another partial cancellation: This one is tricky because there's a half-truth to the cancellation. We'll discuss this particular step in more detail later. The key for now is to recognize how we're still canceling out within an additive term.

$$\frac{3x+1}{x} \overset{\times}{=} \frac{3\not{x}+1}{\not{x}} = 3+1 = 4$$

- Breaking the order of operations: The multiplication being canceled must respect the order of operations. In this example, the 3 and the x should be seen as being grouped together, so that this cancellation cannot happen.

See Section 10 to review this.

$$\frac{3x+1}{x+1} \overset{\times}{=} \frac{3\cancel{x+1}}{\cancel{x+1}} = 3$$

As you continue to move forward into the more algebraically complex world of rational expressions, it's likely that you're going to make some of these errors. The important thing is to not be discouraged. Mistakes are going to happen. The real key is how you respond to the mistakes when you make them. Students often have a reflex of "I forgot" and then quickly move on to the next thing. Unfortunately, this usually does not help them avoid that error in the future. It takes a certain amount of intentional effort and reflection to internalize the algebra.

A helpful recommendation is to keep a record of your algebraic errors. By writing them down, you can start to identify your own patterns of mistakes, which trains your brain to watch out for them in the future. It only takes a few seconds per mistake, but that's sometimes all it takes to to get your brain to start to recognize it.

Factoring Out the Constant (Including the Negative)

Having gone through a list of examples that don't reduce, it's also important to discuss certain types of fractions that do reduce. One type of factorization that students sometimes overlook is to factor out a constant. This can sometimes reveal binomial terms that factor out that are not immediately obvious. Here is an example:

$$\frac{x+2}{4x+8} = \frac{x+2}{4(x+2)} = \frac{\cancel{x+2}}{4\cancel{(x+2)}} = \frac{1}{4}$$

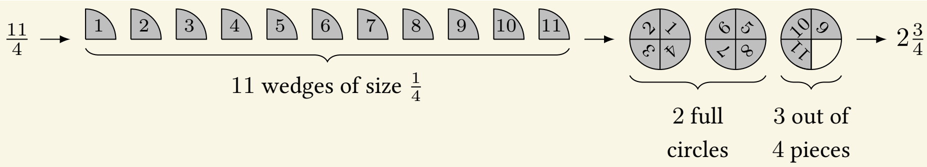
One particular example of this is factoring a negative from the binomial. In general, when it comes to applications involving factoring, we try to keep the coefficient of the x term positive because it helps us to more easily recognize terms that can cancel out. Here is an example of this:

Having a coefficient of 1 for the x term in a binomial is also important for some larger ideas related to polynomials. Specifically, if we do this then it's easy for us to identify the *roots* of the polynomial.

$$\frac{3-x}{x-3} = \frac{-x+3}{x-3} = \frac{-1(x-3)}{x-3} = \frac{-1\cancel{(x-3)}}{\cancel{x-3}} = -1$$

17.9 Solutions to the “Try It” Examples

1



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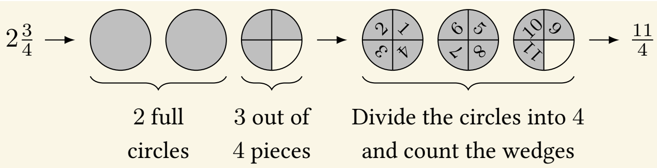

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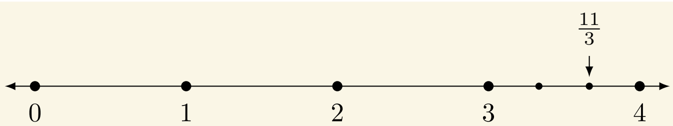


4

$$\frac{6}{8} = \frac{3 \cdot 2}{4 \cdot 2} = \frac{3 \cdot \cancel{2}}{4 \cdot \cancel{2}} = \frac{3}{4}$$

$$\frac{10x^4}{25x^2} = \frac{2x^2 \cdot 5x^2}{5x^2 \cdot 5x^2} = \frac{2x^2 \cdot \cancel{5x^2}}{5x^2 \cdot \cancel{5x^2}} = \frac{2x^2}{5x^2}$$

Find the Common Denominator: Fraction Addition and Subtraction

Learning Objectives:

- ☐ Understand the why a common denominator is important to addition and subtraction of fractions.
- ☐ Identify common multiples and the least common multiple of numbers or variable expressions.
- ☐ Add and subtract fractions with different denominators, including fractions with variables.

Many students will say that they sometimes "forget" how to do arithmetic with fractions. This often results from viewing fractions as purely a collection of symbols with particular rules for how to manipulate them. Their forgetfulness sometimes leads them to guess at how to work with fractions. Here are some examples:

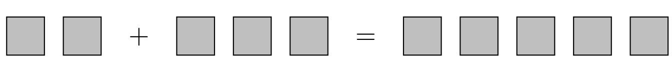
$$\frac{3}{5}+\frac{8}{5}\overset{\times}{=}\frac{3+8}{5+5}=\frac{11}{10} \qquad \frac{2}{5}+\frac{5}{6}\overset{\times}{=}\frac{2+5}{5\cdot 6}=\frac{7}{30}$$

$$\frac{4}{5}-\frac{1}{3}\overset{\times}{=}\frac{4-1}{5-3}=\frac{3}{2} \qquad \frac{3}{5}-\frac{1}{3}\overset{\times}{=}\frac{3-1}{5\cdot 3}=\frac{2}{15}$$

It's not enough to simply identify that these calculations are incorrect. We have been emphasizing the importance of mathematical reasoning throughout this book. Mathematical reasoning goes beyond simply pointing at some rules and saying that they weren't properly followed.

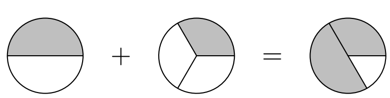
Of course, focusing on a mathematical understanding doesn't mean that incorrect calculations are acceptable.

Let's take a step back and just think about some basic addition and subtraction. What do we mean by $2+3$? There are several ways to think about it, but the one most people think about first is that you are combining two collections of objects:



$$2 \quad + \quad 3 \quad = \quad 5$$

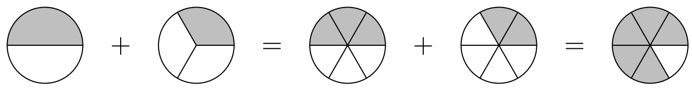
This works well with integers because each piece is the same. But if we try to add fractions, we get a picture that doesn't really make sense.



$$\tfrac{1}{2} \quad + \quad \tfrac{1}{3} \quad = \quad \tfrac{?}{?}$$

No, that last diagram is not $\frac{2}{3}$!

The basic challenge of this is that the pieces are different sizes, so it doesn't really make sense to combine them together like this. And this is the basic explanation of a common denominator. We cannot combine the parts unless they are all the same size.



$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

A “common denominator” is simply a choice of a denominator that works well with the fractions. Mathematically, this means that the chosen denominator is a common multiple of the denominators you’re working with. Ideally, we would use the least common denominator, but if a larger denominator is chosen, the arithmetic will still work out. It would just mean that there may be an extra step of reducing the final answer.

There are many ways to find common multiples of numbers. For fractions involving numbers, we can usually use intuition and experience to get the correct value. But there are also other methods we can use for larger numbers. One of those methods is a “brute force” method where you simply list out multiples of both numbers and look for the smallest number to appear in both lists:

Find the least common multiple of 12 and 32. ⟶

Multiples of 12: $12, 24, 36, 48, 60, 72, 84, \boxed{96}, 108, \ldots$

Multiples of 32: $32, 64, \boxed{96}, 128, 160, \ldots$

This approach works and helps to reinforce the idea of common multiples, but it’s a clumsy approach that doesn’t generalize to variable expressions. A more robust and intuitive approach builds off of finding the greatest common factors.

1 To find the least common multiple of 12 and 32, we will start off by factoring out the greatest common factor from each number:

$$\begin{aligned} 12 &= 4 \cdot 3 \\ 32 &= 4 \cdot 8 \end{aligned}$$

Notice that the common factor is already shared between the two numbers, so we can leave that part alone. The remaining parts are what we might call “unshared” factors. Each part is missing the other’s unshared factor, and so those are the things we need to multiply by to find the least common multiple.

$$\begin{aligned} 12 \cdot 8 &= (4 \cdot 3) \cdot 8 = 96 \\ 32 \cdot 3 &= (4 \cdot 8) \cdot 3 = 96 \end{aligned}$$

The middle step is to help you see the logic of the process.

Try it: Find the least common multiple of 30 and 42.

We’re not going to worry too much about the presentation of this process. This will quickly become a mental calculation.

2 This same approach can be used for finding common multiples of variable expressions. We will use it to find the least common multiple of $9x^2y$ and $15xy^4$.

$$\left.\begin{aligned} 9x^2y &= 3xy \cdot 3x \\ 15xy^4 &= 3xy \cdot 5y^3 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} 9x^2y \cdot 5y^3 &= 45x^2y^4 \\ 15xy^4 \cdot 3x &= 45x^2y^4 \end{aligned}\right.$$

Try it: Find the least common multiple of $4a^2b^3$ and $12a^3b$.

Finding the least common multiple of two expressions is simply a part of adding and subtracting fractions. Once we identify what the least common multiple of the denominators is (the least common denominator), we then need to rewrite both fractions with that denominator, and then add the results.

3 Here is the calculation of $\frac{1}{2}+\frac{1}{3}$:

$$\begin{aligned} \frac{1}{2}+\frac{1}{3} &= \frac{1\cdot 3}{2\cdot 3}+\frac{1\cdot 2}{3\cdot 2} && \text{Common denominator} \\ &= \frac{3}{6}+\frac{2}{6} \\ &= \frac{5}{6} \end{aligned}$$

The last two steps are just arithmetic. We're at the point where we're not going to say anything about it anymore.

Try it: Calculate $\frac{2}{5}+\frac{3}{7}$.

If you are comfortable with these calculations, you can combine the first two steps into one.

4 Subtraction is conceptually different from addition. With subtraction, you are starting with a collection of objects and then taking objects away from that collection. But the same idea for the common denominator still holds. In order for the subtraction to make sense, we need to be working with objects of the same size.

This concept for subtraction works for subtraction involving positive values and giving a positive result. For negatives, things get a little more nuanced.

Try it: Calculate $\frac{5}{4}-\frac{5}{6}$.

5 The same logic can be applied to adding or subtracting fractions with variables in them. Sometimes you will be able to combine like terms, and sometimes you won't. You simply need to pay attention to the information that's in front of you.

$$\begin{aligned} \frac{2x}{y}+\frac{3y}{x^2} &= \frac{2x\cdot x^2}{y\cdot x^2}+\frac{3y\cdot y}{x^2\cdot y} && \text{Common denominator} \\ &= \frac{2x^3}{x^2y}+\frac{3y^2}{x^2y} \\ &= \frac{2x^3+3y^2}{x^2y} \end{aligned}$$

Try it: Calculate $\frac{2x}{3y}+\frac{5}{7x}$.

18.1 Fraction Addition and Subtraction - Worksheet 1

1 Find the least common multiple of 12 and 28 by writing out multiples of each number and also by applying the technique from this section.

2 Find the least common multiple of 18 and 48 by writing out multiples of each number and also by applying the technique from this section.

3 Calculate $\frac{3}{5}+\frac{7}{8}$.

4 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\begin{aligned}\frac{4}{5}+\frac{3}{7} &= \frac{4}{5}\cdot 7+\frac{3}{7}\cdot 5 && \text{Common denominator}\\ &= \frac{28}{35}+\frac{15}{35}\\ &= \frac{43}{35}\end{aligned}$$

This mistake is due to laziness and sloppiness. But neither of those words explain what is wrong.

18.2 Fraction Addition and Subtraction - Worksheet 2

1 Find the least common multiple of 40 and 72.

2 Find the least common multiple of $6x^2y$ and $9xy^3$.

3 Calculate $\frac{11}{6} - \frac{7}{20}$.

4 Calculate $\frac{3}{x} + \frac{7}{y}$.

18.3 Fraction Addition and Subtraction - Worksheet 3

1 Find the least common multiple of 8 and 32.

Students sometimes feel confused by this one. Trust your reasoning skills. If you're still not sure, try it a second way to confirm or reject your first answer.

2 Find the least common multiple of 8 and $3p^2q$.

Trust yourself. Or at least, trust the process.

3 Calculate $\frac{3a}{4b} + \frac{4b}{3a}$.

4 Calculate $\frac{11}{6} + \frac{7}{20}$.

18.4 Fraction Addition and Subtraction - Worksheet 4

1 Calculate $\frac{13}{8} + \frac{17}{20}$.

2 Calculate $\frac{3x}{8y^2} - \frac{5y}{6x}$.

3 Calculate $\frac{5x}{6y^2} + \frac{8}{10x^2y^3}$.

4 Calculate $\frac{x}{y^2} - \frac{y}{x^3}$.

18.5 Fraction Addition and Subtraction - Worksheet 5

1 Calculate $\frac{7}{8} + \frac{5}{24}$.

2 Calculate $\frac{22}{15} - \frac{3}{35}$.

3 Calculate $\frac{a}{b} + \frac{c}{d}$.

4 In the previous problem, you derived a general formula for adding two fractions together. Apply the formula to calculating $\frac{11}{32} + \frac{23}{48}$ and then explain why it's not a good idea to use the formula in every situation.

18.6 Deliberate Practice: Adding and Subtracting Fractions

Focus on these skills:

- Write the original expression.
- Show the multiplication for the common denominator step.
- Try to use the least common denominator rather than applying the general formula.
- Present your work legibly.

Instructions: Perform the indicated calculation..

1. $\frac{7}{12} + \frac{11}{18}$

2. $\frac{13}{10} - \frac{7}{25}$

3. $\frac{5x}{4y} + \frac{8y}{3x}$

4. $\frac{4a}{3b^2} - \frac{2}{7ab}$

5. $\frac{6}{5p^2q} + \frac{2q}{3p}$

6. $\frac{4}{7x^2y} - \frac{3x}{2y^2}$

7. $\frac{3n^2}{m^3} + \frac{4m}{3n^2}$

8. $\frac{4}{xy} + \frac{3}{x^2z}$

9. $\frac{5a}{3b^2c^3} - \frac{4c}{7b^2}$

10. $\frac{3mn}{5p^2} + \frac{5m^2}{8n^2p}$

18.7 Closing Ideas

On the last worksheet, you derived the general formula for adding two fractions. There's a similar one for subtraction.

Definition 18.1. The sum of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is given by the following formula:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

The difference of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is given by the following formula:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

On the very last problem of the worksheet, you also saw the downside to having such a formula. If you had no concept of common denominators, you would have ended up making the problem unnecessarily difficult for yourself because the numbers in your calculation would end up being quite large.

Some students simply learn the "rule" for adding and subtracting fractions with different denominators. These are also the students that tend to forget over time, and they will often start to guess wildly at solutions. Here are some non-examples of adding fractions together:

$$\frac{a}{b} + \frac{c}{d} \overset{\times}{=} \frac{a + c}{b + d} \qquad\qquad \frac{a}{b} + \frac{c}{d} \overset{\times}{=} \frac{a + c}{bd}$$

The major trapping is for students to fall into the habit of simply manipulating the symbols without understanding what's happening. It's very easy to think you understand something because you can execute the proper procedure. But mathematical reasoning goes beyond execution.

One of the reasons this book has short worksheets of problems is because the goal is to have you think more. Yes, practice is important, but those repetitions alone are not the goal.

Take another look at the two errors. Suppose that a friend showed you that work and asked you for help. Would you just tell them that they did it wrong and show them the right way to do it? Would be able to explain the right ideas to them? In the mathematical world, the goal is to both be able to execute the calculation and explain the reasoning behind it. This is a theme that we will keep returning to throughout the book.

18.8 Going Deeper: Adding and Subtracting Rational Expressions

In some ways, the exercises in this section give a false impression of working with rational expressions. It is not often the case that we're adding and subtracting the types of *monomial* expressions that were presented. The reason the exercises look that way is because the emphasis is on learning to recognize when the denominators have common factors and when they don't, which leads to knowing what to multiply by to create the common denominators. By using different variables, it's much easier for students to train themselves to recognize the common factors.

It's likely that you will only ever encounter single variable rational expressions. But the skill of identifying different terms will still come into play because you will need to recognize when *polynomial* factors are the same or different. For example, instead of working with fractions of the form

$$\frac{1}{a} + \frac{1}{b}$$

you will be working with fractions of the form

$$\frac{1}{x+2} + \frac{1}{x-3}.$$

More specifically, the most common manipulation you will encounter will involve binomial factors.

In the previous "Going Deeper" section, we emphasized the importance of multiplicative factors when reducing rational expressions. The same idea holds for adding and subtracting rational expressions. We need to think in terms of multiplicative factors. The way we accomplish this is to think of the whole binomial as a single object. Basically, we want to envision a set of parentheses around each binomial.

$$\frac{1}{x+2} + \frac{1}{x-3} = \frac{1}{(x+2)} + \frac{1}{(x-3)}$$

And once we have done this, all of the experience that we were developing in this section can come into play. All of the previous logic applies to this calculation.

$$\begin{aligned}
\frac{1}{x+2} + \frac{1}{x-3} &= \frac{1\cdot(x-3)}{(x+2)\cdot(x-3)} + \frac{1\cdot(x+2)}{(x-3)\cdot(x+2)} && \text{Common denominator} \\
&= \frac{x-3}{(x+2)(x-3)} + \frac{x+2}{(x+2)(x-3)} \\
&= \frac{(x-3)+(x+2)}{(x+2)(x-3)} \\
&= \frac{2x-1}{(x+2)(x-3)}
\end{aligned}$$

We encourage students to leave the denominator as a product. Since it's easier to multiply than it is to factor, it makes more sense to leave it in the form that is easier to work with.

When subtracting rational expressions, it's extremely important to keep the parentheses around terms when subtracting. This did not come up in the section because we were only working with monomial terms. But when working with binomials, those parentheses are critical to

This idea goes all the way back to Section 3.

avoid errors.

$$\begin{aligned}
\frac{x}{x-1} - \frac{3}{x+3} &= \frac{x \cdot (x+3)}{(x-1)\cdot(x+3)} - \frac{3\cdot(x-1)}{(x+3)\cdot(x-1)} && \text{Common denominator} \\
&= \frac{x^2+3x}{(x-1)(x+3)} - \frac{3x-3}{(x-1)(x+3)} \\
&= \frac{(x^2+3x)-(3x-3)}{(x-1)(x+3)} \\
&= \frac{x^2+3x-3x+3}{(x-1)(x+3)} \\
&= \frac{x^2+3}{(x-1)(x+3)}
\end{aligned}$$

In some expressions, you may find that after adding or subtracting, it may be possible to factor the numerator. When you can see a way to factor, it's best to do it. The reason is that sometimes these factorization steps will reveal that the fraction can be reduced or simplified, which can be very helpful.

$$\begin{aligned}
\frac{1}{x-2} + \frac{x-5}{(x-2)(x+1)} &= \frac{1\cdot(x+1)}{(x-2)\cdot(x+1)} + \frac{x-5}{(x-2)(x+1)} && \text{Common denominator} \\
&= \frac{x+1}{(x-2)(x+1)} + \frac{x-5}{(x-2)(x+1)} \\
&= \frac{(x+1)+(x-5)}{(x-2)(x+1)} \\
&= \frac{2x-4}{(x-2)(x+1)} \\
&= \frac{2(x-2)}{(x-2)(x+1)} \\
&= \frac{2\cdot\cancel{(x-2)}}{(x+1)\cdot\cancel{(x-2)}} \\
&= \frac{2}{x+1}
\end{aligned}$$

It is hard to know in advance which calculations will lead to fractions that reduce and which ones don't, so you will just want to get into the habit of factoring the numerator when you can.

18.9 Solutions to the “Try It” Examples

1

$$\left.\begin{aligned} 30 &= 6 \cdot 5 \\ 42 &= 6 \cdot 7 \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} 30 \cdot 7 &= 210 \\ 42 \cdot 5 &= 210 \end{aligned}\right.$$

2

$$\left.\begin{aligned} 4a^2b^3 &= 4a^2b \cdot b^2 \\ 12a^3b &= 4a^2b \cdot 3a \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} 4a^2b^3 \cdot 3a &= 12a^3b^3 \\ 12a^3b \cdot b^2 &= 12a^3b^3 \end{aligned}\right.$$

3

$$\begin{aligned} \frac{2}{5} + \frac{3}{7} &= \frac{2 \cdot 7}{5 \cdot 7} + \frac{3 \cdot 5}{7 \cdot 5} && \text{Common denominator} \\ &= \frac{14}{35} + \frac{15}{35} \\ &= \frac{29}{35} \end{aligned}$$

4

$$\begin{aligned} \frac{5}{4} - \frac{5}{6} &= \frac{5 \cdot 3}{4 \cdot 3} + \frac{5 \cdot 2}{6 \cdot 2} && \text{Common denominator} \\ &= \frac{15}{12} + \frac{10}{12} \\ &= \frac{5}{12} \end{aligned}$$

5

$$\begin{aligned} \frac{2x}{3y} + \frac{5}{7x} &= \frac{2x \cdot 7x}{3y \cdot 7x} + \frac{5 \cdot 3y}{7x \cdot 3y} && \text{Common denominator} \\ &= \frac{14x^2}{21xy} + \frac{15y}{21xy} \\ &= \frac{14x^2 + 15y}{21xy} \end{aligned}$$

Multiply Straight Across: Fraction Multiplication

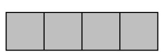
Learning Objectives:

- ☐ Interpret multiplication of fractions as a two step process: Counting the number of wedges and determining the size of the wedges.
- ☐ Multiply fractions involving both numbers and variables.
- ☐ Cancel terms in fraction multiplication problems before multiplying.

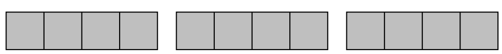
Many students have learned that to multiply fractions, you must "multiply straight across." Not as many students have stopped to ask the question of why that's the right way to multiply fractions. In this section, we're going to explore this idea more deeply to bring some intuition to that manipulation.

We will start by looking at a simple diagram to represent multiplication with integers. One concept we use for multiplication is by having multiple groups with the same number of elements.

Have you noticed the pattern? To understand an idea, we start with simple examples and often draw diagrams.

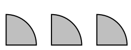
$4 \longrightarrow$ 

4 objects

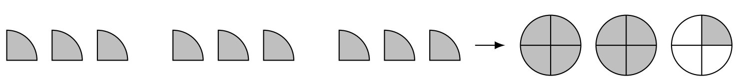
$3 \cdot 4 \longrightarrow$  $\longrightarrow 3 \cdot 4 = 12$

3 groups of 4 objects

The same idea applies to fractions using the wedges from looking at parts of a whole.

$\frac{3}{4} \longrightarrow$ 

3 wedges of size $\frac{1}{4}$

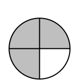
$3 \cdot \frac{3}{4} \longrightarrow$ 

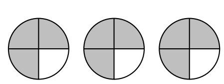
3 groups of 3 wedges of size $\frac{1}{4}$ $\qquad 3 \cdot \frac{3}{4} = \frac{9}{4}$

You could rewrite $\frac{9}{4}$ as $2\frac{1}{4}$, but you should really just get used to seeing improper fractions.

If we think about the structure of a fraction, the numerator represents the number of pieces that we have. And so it would make sense that if we have groups of wedges we would be multiplying the numerator by that quantity.

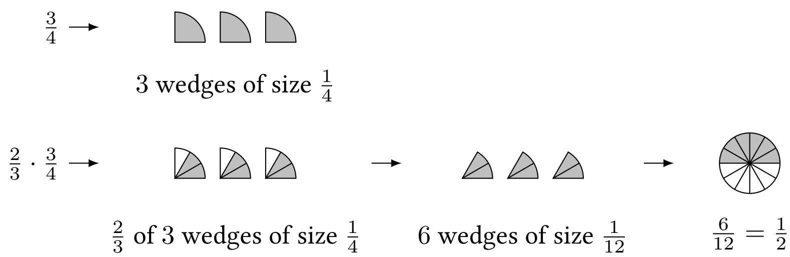
So we have a clear picture of what it looks like to have multiple groupings of fractional quantities. But what about a fractional grouping? We will once again work from diagrams.

$\frac{3}{4}$ of 1 object $\longrightarrow$  $\longrightarrow \frac{3}{4} \cdot 1 = \frac{3}{4}$

$\frac{3}{4}$ of 3 objects $\longrightarrow$  $\longrightarrow \frac{3}{4} \cdot 3 = \frac{9}{4}$

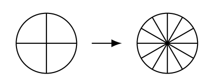
It shouldn't be a surprise that we ended up with the same result since multiplication is commutative. We can even visually see that the two pictures of $\frac{9}{4}$ have the same number of shaded wedges (just in a different arrangement). But this is an important step because it then gives us access to understand multiplying a fraction by a fraction.

See Definition 7.1.



The last set of images gives us the big insight that we need to understand fraction multiplication. We went from $\frac{2}{3}$ of 3 wedges of size $\frac{1}{4}$ to 6 wedges of size $\frac{1}{12}$. Let's take a look at each of these parts separately.

- 6 wedges: We started with 3 wedges, and each of these turned into 2 (smaller) wedges. So we have that the 6 comes from 3 groups of 2 wedges.
- Wedges of size $\frac{1}{12}$: The starting wedge size was $\frac{1}{4}$, which corresponds to cutting the whole into 4 equal pieces. Each of these pieces was cut into 3 pieces, which gives a total of 12 pieces for the whole. This is why that the wedges are size $\frac{1}{12}$.



Cut each wedge into 3 pieces.

This observation shows us that fraction multiplication is essentially a two-step process. We multiply the numerators together to get an accurate count of the number of wedges, and we multiply the denominators together to get the correct wedge size.

1 Now that we have the conceptual framework in place, we can understand the why the phrase "multiply straight across" for fraction multiplication actually works. For a fraction, the numerator represents the number of wedges and the denominator represents the size of the wedges. When we multiply the numerators, we're counting for the total number of wedges (a number of groups of a certain size). When we multiply the denominators, we're establishing the size of the wedges (based on parts of a part of a whole).

Try it: Draw a diagram to represent the product $\frac{1}{2} \cdot \frac{3}{4}$.

As you draw the diagrams, make sure to think through the meaning.

The practice of multiplying fractions looks nothing like the concept. This is not an uncommon situation in life. Most adults know how to drive a car, but very few of them can actually explain the processes involved between pressing the gas pedal and the car moving. In this analogy, we've just finished describing how the engine works, and now we're going to just focus on driving the car.

Can you think of other examples?

The "rule" of multiplying straight across is expressed using algebraic symbols in the following manner.

Definition 19.1. The product of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ is given by the following formula:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

The idea of multiplying straight across applies regardless of whether we're working with numerical fractions or fractions involving variables.

2

Try it: Calculate $\frac{2}{5} \cdot \frac{6}{7}$ and $\frac{3x}{4} \cdot \frac{7x}{2}$.

This is one of the few times in this book where you're just plugging things into a formula.

3

In some situations, you will find that multiplying straight across will result in very large numbers.

$$\frac{32}{21} \cdot \frac{35}{24} = \frac{1120}{504}$$

It turns out that by using a little foresight, large fractions can sometimes be avoided by reducing before multiplying. The idea is the same as reducing a fraction. We are going to look for factors that appear in the product for both the numerator and the denominator. This requires you to identify common factors between the terms in the numerator and the terms in the denominator.

$$\begin{aligned} \frac{32}{21} \cdot \frac{35}{24} &= \frac{4 \cdot 8}{3 \cdot 7} \cdot \frac{5 \cdot 7}{3 \cdot 8} && \text{Identify common factors} \\ &= \frac{4 \cdot \cancel{8}}{3 \cdot \cancel{7}} \cdot \frac{5 \cdot \cancel{7}}{3 \cdot \cancel{8}} && \text{Reduce} \\ &= \frac{4}{3} \cdot \frac{5}{3} \\ &= \frac{20}{9} \end{aligned}$$

Try it: Calculate $\frac{12}{25} \cdot \frac{40}{9}$ by reducing before multiplying.

This is similar to the situation on the last worksheet for fraction addition and subtraction.

4

The exact same idea can be applied to fractions with variables.

$$\begin{aligned} \frac{4x}{9y^3} \cdot \frac{15y^2}{2x^4} &= \frac{2 \cdot 2x}{3y \cdot 3y^2} \cdot \frac{5 \cdot 3y^2}{x^3 \cdot 2x} && \text{Identify common factors} \\ &= \frac{2 \cdot \cancel{2x}}{3y \cdot \cancel{3y^2}} \cdot \frac{5 \cdot \cancel{3y^2}}{x^3 \cdot \cancel{2x}} && \text{Reduce} \\ &= \frac{2}{3y} \cdot \frac{5}{x^3} \\ &= \frac{10}{3x^3y} \end{aligned}$$

Try it: Calculate $\frac{6t}{5} \cdot \frac{8}{3t^3}$ by reducing before multiplying.

19.1 Fraction Multiplication - Worksheet 1

1

Draw a diagram to represent the product $\frac{1}{4} \cdot \frac{3}{2}$.

2

Draw a diagram to represent the product $\frac{2}{3} \cdot \frac{3}{4}$.

3

Calculate $\frac{4}{5} \cdot \frac{8}{3}$.

4

Calculate $\frac{4}{5} \cdot \frac{8}{3}$.

19.2 Fraction Multiplication - Worksheet 2

1 Draw a diagram to represent the product $\frac{3}{4} \cdot \frac{3}{4}$.

2 Calculate $\frac{10}{9} \cdot \frac{4}{7}$.

3 Calculate $\frac{12}{5} \cdot \frac{15}{8}$.

4 Calculate $\frac{3a}{8} \cdot \frac{5}{7b}$.

19.3 Fraction Multiplication - Worksheet 3

1 Calculate $\frac{3}{11} \cdot \frac{8}{5}$.

2 Calculate $\frac{25}{6} \cdot \frac{8}{3}$.

3 Calculate $\frac{5x}{8y} \cdot \frac{7x^2}{9}$.

4 Calculate $\frac{15p^2}{7q} \cdot \frac{21pq}{5}$.

19.4 Fraction Multiplication - Worksheet 4

1 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\frac{3}{4} \cdot \frac{4}{3} = \frac{\not{3}}{\not{4}} \cdot \frac{\not{4}}{\not{3}} \qquad \text{Reduce}$$
$$= 0$$

2 Calculate $\frac{3}{5} \cdot 5x$.

Can you explain why $5x = \frac{5x}{1}$? Can you think of another way of understanding this calculation?

3 Calculate $\frac{15x^2y}{8z} \cdot \frac{10xz^2}{9y}$.

4 Calculate $\frac{5(x+2)^2}{8(x-3)} \cdot \frac{4(x-3)^3}{15(x+2)^4}$.

Keep the terms in the parentheses together. Do not use the distributive property to expand those terms.

19.5 Fraction Multiplication - Worksheet 5

1 Check the presentation for errors. If you find one, circle it and describe the mistake in words.

$$\frac{5}{14}\cdot\frac{7}{15} = \frac{5}{2\cdot 7}\cdot\frac{7}{3\cdot 5} \qquad \text{Identify common factors}$$

$$= \frac{\not{5}}{2\cdot \not{7}}\cdot\frac{\not{7}}{3\cdot \not{5}} \qquad \text{Reduce}$$

$$= 6$$

2 Calculate $\frac{15p^2}{7q}\cdot\frac{21pq}{5}$.

3 Calculate $\frac{x^2-1}{x^2-x-6}\cdot\frac{x^2-5x+6}{x+1}$.

Factor the numerator and the denominator.

19.6 Deliberate Practice: Multiplying Fractions

Focus on these skills:

- Write the original expression.
- Show the factoring and cancellation step.
- Present your work legibly.

Instructions: Perform the indicated calculation.

1. $$\frac{48}{35} \cdot \frac{25}{36}$$

2. $$\frac{20}{27} \cdot \frac{18}{25}$$

3. $$\frac{45}{7} \cdot \frac{28}{15}$$

4. $$\frac{15}{4ab^2} \cdot \frac{8a^2}{5b}$$

5. $$\frac{14x^2}{9y^3} \cdot \frac{15xy^2}{8}$$

6. $$\frac{9n^2m^3}{5} \cdot \frac{20nm^3}{3}$$

7. $$\frac{15xz^2}{4y^3} \cdot \frac{2yz}{21x^4}$$

8. $$\frac{14a^2}{3b^2c^3} \cdot \frac{15a^2c}{7b^3}$$

9. $$\frac{2x^2(x-3)}{3(x-4)^2} \cdot \frac{6x(x-4)}{5(x-3)^2}$$

10. $$\frac{12(x+2)^2(x-1)}{5(x+3)^2} \cdot \frac{10(x+2)}{9(x-1)(x+3)^3}$$

19.7 Going Deeper: Additional Factoring Techniques

In Section 9 and its "Going Deeper" section, we looked at various factoring ideas and techniques. We're going to push that a little bit deeper here. Before getting started, we want to remind ourselves of the most common special factorizations patterns from that section:

Square of a binomial sum:	$a^2 + 2ab + b^2 = (a + b)^2$
Square of a binomial difference:	$a^2 - 2ab + b^2 = (a - b)^2$
Difference of squares:	$a^2 - b^2 = (a + b)(a - b)$
Sum of cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

When working with these expressions, we noted how we can use these formulas to factor expressions even when the expressions contain multiple variables or higher powers of the variables. As long as the expression can be made to fit the form, then the factorization works.

But there's something else that can happen with those more complicated expressions, which is that it may take multiple factorizations to factor it completely. In other words, after one factorization step, you may discover that the factors can be factored again. Here is an example:

It's the same as larger numbers having more factors than smaller numbers.

$$\begin{aligned} x^4 - 1 &= (x^2 + 1)(x^2 - 1) && \text{Difference of squares} \\ &= (x^2 + 1)(x + 1)(x - 1) && \text{Difference of squares} \end{aligned}$$

Unfortunately, there is no indicator that you will need to continue to factor other than recognizing that you can continue to factor. However, as you gain experience, you will get better at recognizing when expressions can be factored and when they can't.

It turns out that the ac method of factoring can also be extended to factor certain types of expressions. Expressions of this type are said to be *quadratic in form*, which means that we can treat them as a quadratic expression by thinking about them in the right way. Here is an example:

$$x^4 + 4x^2 - 5$$

This is not a quadratic expression, but it said to be *quadratic in form*. This means that if we think about it the right way, we can treat it like a quadratic expression. Specifically, we can rewrite it by treating x^2 as the variable:

$$\begin{aligned} x^4 + 4x^2 - 5 &= (x^2)^2 + 4(x^2) - 5 && \text{Rewrite as a quadratic} \\ &= (x^2 + 5)(x^2 - 1) && ac \text{ method} \\ &= (x^2 + 5)(x + 1)(x - 1) && \text{Difference of squares} \end{aligned}$$

Some students find it easier to substitute $y = x^2$ so that this becomes $y^2 + 4y - 5$ before attempting to factor it.

It would be nice if there were some fixed set of rules for when to factor expressions and when to leave them alone. Unfortunately, those decisions are often driven by context. The most common place this becomes an issue is with a difference of squares. For example, consider this quadratic expression:

$$x^2 - 3$$

On the one hand, we can look at this and say that 3 is not a perfect square and just leave it in that form. On the other hand, while 3 isn't a perfect square, it is the square of $\sqrt{3}$, which means that we can still factor it.

A *perfect square* is the square of an integer.

$$\begin{aligned} x^2 - 3 &= x^2 - (\sqrt{3})^2 && \text{Rewrite 3 as a square} \\ &= (x + \sqrt{3})(x - \sqrt{3}) && \text{Difference of squares} \end{aligned}$$

Does this mean that we should factor $x^2 - 3$ using square roots? The answer is that it depends. The factorization into linear terms is helpful for some mathematical operations, but other times it's an unnecessary step. So the decision of whether to factor will be dependent upon the context. This is an idea that we've touched on many times before, which is that you don't want to think about math as a set of rules that you follow every single time. It is better to recognize that you have the option of taking that factorization one step further, but then to decide what you're going to do based on the specific goals for the problem.

This may not be the last factoring technique that you'll encounter. In many precalculus courses, you will learn about other techniques that can be applied to help to sometimes factor higher degree polynomials that don't fit any of our current ideas. But even with that technique, you still will not be able to factor everything. In fact, one of the big questions that mathematicians had wondered about for a long time is whether there's a formula that can be used to factor everything. This question was ultimately answered in the early 1800s, when it was proven that there are fifth degree equations that have roots that are impossible write down.

Students usually learn a combination of the rational root theorem and either synthetic division or polynomial long division.

This question is answered by a mathematical field known as Galois Theory.

19.8 Solutions to the "Try It" Examples

1

$\frac{1}{2} \cdot \frac{3}{4} \rightarrow$

$\frac{1}{2}$ of 3 wedges of size $\frac{1}{4}$ 3 wedges of size $\frac{1}{8}$ $\frac{3}{8}$

2

$$\begin{aligned}\frac{2}{5}\cdot\frac{6}{7} &= \frac{2\cdot 6}{5\cdot 7}\\ &= \frac{12}{35}\end{aligned}$$

$$\begin{aligned}\frac{3x}{4}\cdot\frac{7x}{2} &= \frac{3x\cdot 7x}{4\cdot 2}\\ &= \frac{21x^2}{8}\end{aligned}$$

3

$$\begin{aligned}\frac{12}{25}\cdot\frac{40}{9} &= \frac{4\cdot 3}{5\cdot 5}\cdot\frac{8\cdot 5}{3\cdot 3} && \text{Identify common factors}\\ &= \frac{4\cdot \cancel{3}}{5\cdot \cancel{5}}\cdot\frac{8\cdot \cancel{5}}{3\cdot \cancel{3}} && \text{Reduce}\\ &= \frac{4}{5}\cdot\frac{8}{3}\\ &= \frac{32}{15}\end{aligned}$$

4

$$\begin{aligned}\frac{6t}{5}\cdot\frac{8}{3t^3} &= \frac{2\cdot 3t}{5}\cdot\frac{8}{t^2\cdot 3t} && \text{Identify common factors}\\ &= \frac{2\cdot \cancel{3t}}{5}\cdot\frac{8}{t^2\cdot \cancel{3t}} && \text{Reduce}\\ &= \frac{2}{5}\cdot\frac{8}{t^2}\\ &= \frac{16}{5t^2}\end{aligned}$$

Multiply by the Reciprocal: Fraction Division

Learning Objectives:

- ☐ Relate division to multiplication both conceptually and algebraically.
- ☐ Divide fractions involving both numbers and variables.

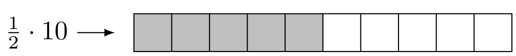
Consider the following story:

> Suppose there are ten pieces of candy in the candy jar. You and a friend decide that you'll each take one half of the candy. So you divide the candy into two equal portions, and you each enjoyed your share of five pieces of candy.

It not a particularly interesting story from the storytelling perspective. But something very interesting happened from a mathematical point of view. Let's take a closer look.

In one way of looking at the story, we're talking about a multiplication problem. Both you and your friend each get half of a group of 10.

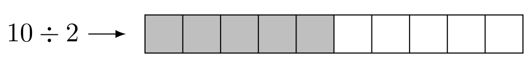
10 objects



$\frac{1}{2}$ of a group of 10 objects

On the other hand, this is a story about dividing a collection into two equal parts.

10 objects



10 objects divided into 2 equal groups

What this is showing us is that there is a relationship between multiplication and division. Multiplying by $\frac{1}{2}$ is the same as dividing by 2. And while this is a simple example, it does help us to clearly see that there is a relationship. We can even write 2 as $\frac{2}{1}$ (since $2 \div 1 = 2$) and use this to declare that dividing is the same as multiplying by the reciprocal of a number. But this doesn't actually explain anything. It just observes a computation and then declares it to be a rule without explaining bringing any insight as to why. This is another example of the difference between driving a car and understanding how it works.

The formal definition of the reciprocal is below.

"I step on the pedal and it goes" is not an explanation of how a car works.

There are many levels of understanding fraction division. We're going to focus on the algebraic understanding here, and give you a chance to explore some of the more visual interpretations in the worksheets. Let's take a look at a general division of fractions calculation: $\frac{a}{b} \div \frac{c}{d}$. Since fractions are a representation of division, we could rewrite this as a fraction where the numerator and denominator are themselves fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

At first, the may appear to be a worse situation, since the notation seems to be a mess of symbols. But when we write it this way, we can use what we know about rewriting fractions, which is that as long as we multiply or divide the top and bottom of a fraction by the same quantity, we do not change its value. With a little bit of insight, we can come to the conclusion that our goal is to make it so that the numerator and denominator of the overall fraction are just integers.

We did this when found common denominators for adding and subtracting fractions.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot bd}{\frac{c}{d} \cdot bd} = \frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$$

Notice that we have rewritten the division calculation as a multiplication calculation, and that the second fraction got flipped over. This is why some students learn fraction division as "keep-change-flip." But this pushes the calculation even deeper into the realm of simply following rules. So we will talk about this at a higher mathematical level by using proper terminology.

Definition 20.1. The *reciprocal* of the fraction $\frac{a}{b}$ is $\frac{b}{a}$ as long as $a \neq 0$. If $a = 0$, then we say that the reciprocal does not exist.

From this definition, we can say that dividing by a fraction is the same as multiplying by the reciprocal. And we can see that this is derived from writing division as a fraction and simplifying the fraction.

1

Try it: Calculate $\frac{3}{7} \div \frac{9}{2}$.

2

Sometimes, the problem will be given to you as fractions inside of fractions. At that point, you can go either multiply by the reciprocal or simplify the fraction by multiplying the top and the bottom by the same quantity.

In fact, you can also turn problems written with the ÷ symbol as a fraction of fractions then use either of these techniques.

This is what multiplication by the reciprocal looks like:

$$\begin{aligned} \frac{\frac{6}{25}}{\frac{7}{3}} &= \frac{6}{25} \cdot \frac{3}{7} && \text{Multiply by the reciprocal} \\ &= \frac{18}{175} \end{aligned}$$

And this is what simplifying within the fraction looks like:

$$\begin{aligned} \frac{\frac{6}{25}}{\frac{7}{3}} &= \frac{\frac{6}{\cancel{25}} \cdot \cancel{25} \cdot 3}{\frac{7}{\cancel{3}} \cdot 25 \cdot \cancel{3}} && \text{Simplify the fraction} \\ &= \frac{18}{175} \end{aligned}$$

This is the same step as multiplying the top and bottom by the same number when adding or subtracting fractions.

It is helpful to be familiar with both of these. As fractions become more complicated, there are times that multiplying by the reciprocal is the more difficult approach. Ultimately, those are decisions you will learn to make based on your experience and the specific expression you're working with.

Try it: Calculate $\frac{\frac{2}{3}}{\frac{5}{14}}$.

Just as with the previous section, we can reduce before multiplying in order to simplify our calculations, and we can also use this with fractions involving variables.

3

Try it: Calculate $\frac{6}{5} \div \frac{18}{25}$.

4

Try it: Calculate $\frac{8x}{15y^2} \div \frac{4x^2y}{9}$.

20.1 Fraction Division - Worksheet 1

1 Calculate $\frac{4}{15} \div \frac{8}{3}$ by rewriting the calculation as multiplication by the reciprocal.

2 Calculate $\frac{\frac{4}{15}}{\frac{8}{3}}$ by rewriting the calculation as multiplication by the reciprocal.

3 Calculate $\frac{\frac{4}{15}}{\frac{8}{3}}$ by multiplying the numerator and denominator by the same value to simplify the fraction.

4 All three of these problems were the same underlying calculation, but presented and worked out in three separate ways. Which way makes the most intuitive sense to you?

Even though one way may more intuitive for you, it is helpful to understand and practice all three of these methods.

20.2 Fraction Division - Worksheet 2

1 Calculate $\frac{30}{11} \div \frac{18}{55}$.

2 Calculate $\frac{21}{10} \div \frac{45}{16}$.

3 Calculate $\frac{9a^2b}{14} \div \frac{12b}{35a}$.

4 Calculate $\frac{15y}{8x^2} \div \frac{9y^2}{20x}$.

20.3 Fraction Division - Worksheet 3

1 Calculate $\frac{12}{11} \div \frac{21}{44}$.

2 Calculate $\frac{18}{25} \div \frac{14}{15}$.

3 Calculate $\frac{27p^2}{20q} \div \frac{45p}{28q^3}$.

4 Calculate $\frac{16x^3}{25y^2} \div \frac{6y}{5x}$.

20.4 Fraction Division - Worksheet 4

1 We are going to work with one interpretation of fraction division that uses common denominators. The fraction $\frac{a}{b}$ can be interpreted as having a wedges of size $\frac{1}{b}$. This means that the division calculation $\frac{16}{3} \div \frac{4}{3}$ can be understood as having 16 wedges of size $\frac{1}{3}$ and creating groups of 4 wedges of size $\frac{1}{3}$. With this framework in mind, draw a diagram of 16 wedges of size $\frac{1}{3}$ and divide it into groups of 4 wedges each. Use this to determine the result of the calculation.

2 Using algebraic methods, verify that $\frac{a}{c} \div \frac{b}{c} = \frac{a}{b}$.

3 Calculate $\frac{12}{5} \div \frac{4}{3}$ using a common denominator.

20.5 Fraction Division - Worksheet 5

1 Calculate $\frac{5}{4} \div \frac{2}{3}$ using a common denominator.

2 Calculate $\frac{25}{6} \div \frac{5}{4}$ using a common denominator.

3 Calculate $\frac{7}{16} \div \frac{5}{24}$ using a common denominator.

4 Calculate $\frac{12x}{5y^2} \div \frac{6xy}{35}$ using a common denominator.

20.6 Deliberate Practice: Dividing Fractions

Focus on these skills:

- Write the original expression.
- Show the factoring and cancellation step.
- Present your work legibly.

Instructions: Perform the indicated calculation.

1. $\dfrac{36}{35} \div \dfrac{21}{25}$

2. $\dfrac{8}{15} \div \dfrac{20}{27}$

3. $\dfrac{45}{28} \div \dfrac{15}{14}$

4. $\dfrac{9x^2}{5y^3} \div \dfrac{6xy^2}{35}$

5. $\dfrac{15b}{4a^2} \div \dfrac{9a^2b^2}{10}$

6. $\dfrac{9}{5n^2m^3} \div \dfrac{3nm^2}{20}$

7. $\dfrac{8xy^2}{15z^3} \div \dfrac{2yz^2}{21x^4}$

8. $\dfrac{14b^2}{3a^3c^3} \div \dfrac{28a^2b^3}{15c^3}$

9. $\dfrac{2x^2(x+3)}{3(x-4)^2} \div \dfrac{4(x+3)}{15x(x-4)^2}$

10. $\dfrac{15(x-3)^2(x+1)}{9(x+3)^2} \div \dfrac{10(x+1)(x-3)}{3(x+3)^3}$

20.7 Closing Ideas

We opened this section with the idea that multiplication and division were related by thinking about how taking 10 objects and splitting it into two equal groups looks the same as taking half of a group of 10 objects. But we didn't elaborate on the nature of that relationship.

Multiplication and division are known as inverse operations. Basically, it means that one operation undoes the other. We've actually already seen this idea, but without those words. When we were solving equations, we would run into situations that look like the following:

$$\begin{aligned} 3x &= 12 & \\ x &= 4 & \text{Divide both sides by 3} \end{aligned}$$

The act of dividing both sides by 3 is undoing the multiplication of x by 3 on the left side of the equation. In fact, we can do the same thing when solving equations that involve fractions.

$$\begin{aligned} \frac{x}{3} &= 4 & \\ x &= 12 & \text{Multiply both sides by 3} \end{aligned}$$

The idea that multiplication is the inverse of division and that division is multiplication by the reciprocal has another important parallel. We run into the exact same situation with addition and subtraction. When solving equations, you have to subtract to undo addition and you have to add to undo subtraction.

This is the algebraic representation of subtraction as the addition of the opposite:

$$a - b = a + (-b)$$

$$\begin{aligned} x - 4 &= 5 & \\ x &= 9 & \text{Add 4 to both sides} \end{aligned}$$

$$\begin{aligned} x + 4 &= 9 & \\ x &= 5 & \text{Subtract 4 from both sides} \end{aligned}$$

It is often said that subtraction is addition of the opposite. This phrase is very close to our division phrase. We will put these phrases next to each other to see the comparison.

Division	is	multiplication	by	the reciprocal.
Subtraction	is	addition	of	the opposite.

This shows us that addition and multiplication are the fundamental operations on numbers. In some ways, this may help to explain why addition is easier than subtraction and why multiplication is easier than division. Some operations are just more basic and more fundamental than others. These ideas are also at the core of an area of mathematics known as *field theory*, which is built on many of the ideas that we've already encountered. So it turns out that high level mathematics has its roots in things that we teach to all students.

20.8 Going Deeper: Fractions Inside of Fractions

The technique of multiplying the top and bottom of a fraction by the same value to eliminate the denominators seems odd when working with fractions involving numbers, but its value goes up significantly when working with rational expressions. One of the more algebraically difficult situations to work with is when you have fractions inside of other fractions. It's not that anything about the process is different, but the sheer number of symbols can cause students to feel overwhelmed and make mistakes.

Here is an example. The goal is to simplify the following fraction:

$$\frac{\frac{1}{x+1} - \frac{3}{x-2}}{\frac{2}{x-2} + \frac{1}{x+3}}$$

We will first do the calculation by adding the fractions in the numerator and denominator, and then dividing the resulting fractions by multiplying by the reciprocal.

$$\begin{aligned}
\frac{\frac{1}{x+1} - \frac{3}{x-2}}{\frac{2}{x-2} + \frac{1}{x+3}} &= \frac{\frac{1\cdot(x-2)}{(x+1)\cdot(x-2)} - \frac{3\cdot(x+1)}{(x-2)\cdot(x+1)}}{\frac{2\cdot(x+3)}{(x-2)\cdot(x+3)} + \frac{1\cdot(x-2)}{(x+3)\cdot(x-2)}} && \text{Common Denominator} \\
&= \frac{\frac{(x-2)-(3x+3)}{(x+1)(x-2)}}{\frac{(2x+6)+(x-2)}{(x-2)(x+3)}} \\
&= \frac{\frac{-2x-5}{(x+1)(x-2)}}{\frac{3x+4}{(x-2)(x+3)}} && \text{Combine like terms} \\
&= \frac{-2x-5}{(x+1)(x-2)} \cdot \frac{(x-2)(x+3)}{3x+4} && \text{Multiply by the reciprocal} \\
&= -\frac{(2x+5)(x-2)(x+3)}{(x+1)(x-2)(3x+4)} && \text{Multiply and factor out the negative} \\
&= -\frac{(2x+5)(x+3)\cdot \cancel{(x-2)}}{(x+1)(3x+4)\cdot \cancel{(x-2)}} && \text{Reduce} \\
&= -\frac{(2x+5)(x+3)}{(x+1)(3x+4)} && \text{Reduce}
\end{aligned}$$

We will now do the same calculation, but we will multiply the numerator and denominator

This process somewhat mirrors the process of clearing the denominator in the "Going Deeper" portion of Section 1.

by the same value to eliminate the fractions inside of the fraction.

$$\begin{aligned}
\frac{\frac{1}{x+1}-\frac{3}{x-2}}{\frac{2}{x-2}+\frac{1}{x+3}} &= \frac{\frac{1}{x+1}-\frac{3}{x-2}}{\frac{2}{x-2}+\frac{1}{x+3}} \cdot \frac{(x+1)(x-2)(x+3)}{(x+1)(x-2)(x+3)} && \text{Multiply top and bottom} \\
&= \frac{\frac{(x+1)(x-2)(x+3)}{x+1}-\frac{3(x+1)(x-2)(x+3)}{x-2}}{\frac{2(x+1)(x-2)(x+3)}{x-2}+\frac{(x+1)(x-2)(x+3)}{x+3}} && \text{Distribute} \\
&= \frac{\frac{\cancel{(x+1)}(x-2)(x+3)}{\cancel{x+1}}-\frac{3(x+1)\cancel{(x-2)}(x+3)}{\cancel{x-2}}}{\frac{2(x+1)\cancel{(x-2)}(x+3)}{\cancel{x-2}}+\frac{(x+1)(x-2)\cancel{(x+3)}}{\cancel{x+3}}} && \text{Reduce} \\
&= \frac{(x-2)(x+3)-3(x+1)(x+3)}{2(x+1)(x+3)+(x+1)(x-2)} && \text{Rewrite and simplify} \\
&= \frac{(x^2+x-6)-(3x^2+12x+9)}{(2x^2+8x+6)+(x^2-x-2)} \\
&= \frac{-2x^2-11x-15}{3x^2+7x+4} \\
&= -\frac{2x^2+11x+15}{3x^2+7x+4} && \text{Factor out the negative} \\
&= -\frac{(2x+5)(x+3)}{(x+1)(3x+4)}
\end{aligned}$$

Both of these calculations involve a lot of steps, and neither one is necessarily better than the other. The first one required the proper execution of more fraction calculations, but the second one required a factorization step to ensure that the expression cannot be reduced. Both of those steps can be difficult in their own ways.

You should go through each calculation step by step to make sure that you really understand what happened.

It should be noted that expressions that are this messy do not appear particularly frequently, but if you continue along in mathematics, you will find that calculations involving fractions inside of fractions do appear. There are two reasons for including this calculation here. The first is to expose you to some of the more intricate algebraic manipulations that arise in higher levels of mathematics. The second is to point out the fact that even though the algebra may look complicated, there's nothing here that you haven't seen before or is beyond your ability. Your capacity for handling complicated expressions like these will grow as you continue to practice and gain experience.

20.9 Solutions to the "Try It" Examples

1

$$\begin{aligned}\frac{3}{7} \div \frac{9}{2} &= \frac{3}{7} \cdot \frac{2}{9} && \text{Multiply by the reciprocal}\\ &= \frac{1 \cdot \cancel{3}}{7} \cdot \frac{2}{3 \cdot \cancel{3}} && \text{Reduce}\\ &= \frac{1}{7} \cdot \frac{2}{3}\\ &= \frac{2}{21}\end{aligned}$$

2

$$\begin{aligned}\frac{\frac{2}{3}}{\frac{5}{14}} &= \frac{\frac{2}{\cancel{3}} \cdot \cancel{3} \cdot 14}{\frac{5}{\cancel{14}} \cdot 3 \cdot \cancel{14}} && \text{Simplify the fraction}\\ &= \frac{28}{15}\end{aligned}$$

3

$$\begin{aligned}\frac{6}{5} \div \frac{18}{25} &= \frac{6}{5} \cdot \frac{25}{18} && \text{Multiply by the reciprocal}\\ &= \frac{1 \cdot \cancel{6}}{1 \cdot \cancel{5}} \cdot \frac{5 \cdot \cancel{5}}{3 \cdot \cancel{6}} && \text{Reduce}\\ &= \frac{1}{1} \cdot \frac{5}{3}\\ &= \frac{5}{3}\end{aligned}$$

4

$$\begin{aligned}\frac{8x}{15y^2} \div \frac{4x^2y}{9} &= \frac{8x}{15y^2} \cdot \frac{9}{4x^2y} && \text{Multiply by the reciprocal}\\ &= \frac{2 \cdot \cancel{4x}}{5y^2 \cdot \cancel{3}} \cdot \frac{3 \cdot \cancel{3}}{xy \cdot \cancel{4x}} && \text{Reduce}\\ &= \frac{2}{5y^2} \cdot \frac{3}{xy}\\ &= \frac{6}{5xy^3}\end{aligned}$$

Making Sense with Dollars and Cents: Decimal Addition and Subtraction

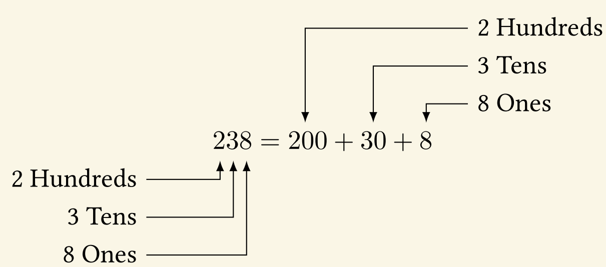
Learning Objectives:

- ☐ Understand the relationship between decimals and fractions.
- ☐ Add and subtract decimals.

Although fractions are important and useful for algebraic manipulations, many of the day-to-day numbers that people encounter are decimals. But it turns out that decimals are just a special way of writing parts of a whole, which means that they're also just fractions. Most students don't have a good sense of the interplay between the two notations, and quite often think of them as mostly unrelated to each other.

Where do decimals even come from? It turns out that this is an extension of the way we write numbers. We use what a *place value system* which means that our numbers depend on both the symbol we use (the digits 0-9) and the position of that digit within the number itself.

1 Let's think about the number 237. We have been trained to understand that when we put digits side-by-side like this to make larger numbers, that each digit refers to a different group size.



The normal way of writing numbers is known as the *standard form* of a number.

Breaking a number into distinct units like this is called the *expanded form* of a number.

Try it: Write the number 8367 in expanded form and label each of the parts with the corresponding unit (as shown above).

The choice to use ones, tens, and hundreds (and also thousands, ten thousands, and so forth) is because each of those numbers turns out to be a power of 10.

Units	Representations
Ones	$10^0 = 1$
Tens	$10^1 = 10$
Hundreds	$10^2 = 100$
Thousands	$10^3 = 1\,000$
Ten thousands	$10^4 = 10\,000$
Hundred thousands	$10^5 = 100\,000$
Millions	$10^6 = 1\,000\,000$
Billions	$10^9 = 1\,000\,000\,000$
Trillions	$10^{12} = 1\,000\,000\,000\,000$

For the sake of space, we skipped over ten million and a hundred million, and similarly for billions. It just follows the same pattern.

Notice that as we go down the list, the exponents increase and so do the numbers. If we go up the list, the exponents get decrease and so do the numbers. And in the same way we used this pattern to develop negative exponents, we can develop decimals.

Units	Representations
Thousandths	$\frac{1}{10^3} = 10^{-3} = 0.001$
Hundredths	$\frac{1}{10^2} = 10^{-2} = 0.01$
Tenths	$\frac{1}{10^1} = 10^{-1} = 0.1$
Ones	$10^0 = 1$
Tens	$10^1 = 10$
Hundreds	$10^2 = 100$
Thousands	$10^3 = 1\,000$

The naming of these units are a little awkward to say out loud, but the mathematics behind them is simple. Tenths are the size you get when you take one object and break it into ten pieces. Hundredths are the size you get when you take one object and break it into one hundred pieces.

2

Try it: Write the number 35.79 **in expanded form and label each of the parts with the corresponding unit.**

It turns out that decimals have multiple representations. This comes out of thinking about multiple representations of the same fraction.

$$\begin{aligned} 0.3 = \frac{3}{10} &= \frac{30}{100} = 0.30 \\ &= \frac{300}{1000} = 0.300 \\ &= \frac{3000}{10000} = 0.3000 \end{aligned}$$

3

There is a simple connection between the number of decimals and the powers of ten. The number of zeros in the power of 10 in the denominator corresponds to the number of decimal places for that representation of the number. For example, the number 0.0365 (four decimal places) corresponds to $\frac{365}{10000}$ (four zeros in the denominator). In fact, this number also corresponds to the exponent of the 10 in the denominator: $\frac{365}{10000} = \frac{365}{10^4}$. Thinking about numbers this way will help to avoid certain types of errors.

Try it: Write the number 0.086 **as a fraction.**

For the purposes of this section, it's not important to reduce the fraction.

Most of the challenges with addition and subtraction of decimals are resolved by simply ensuring that all your numbers have the same number of decimal places. The best analogy for this is money. American currency is always written with two decimal places (if decimal places are being used). And what this does is that it helps us think about the coins relative to same base unit (1 cent) all the time, and there's no confusion about whether a dime ($0.10) is the same as a penny ($0.01).

If all the numbers have the same number of decimals, it's ensuring that all of the corresponding fractions have the same denominator.

4 It can be helpful to ensure that all of the numbers are written to the same number of decimals if the calculation is intended to be performed mentally or by hand. This will help to reinforce the underlying concept as well as avoid computational errors.

$$3.04 + 1.1 = 3.04 + 1.10 = 4.14$$

Try it: Calculate $2.5 + 1.22$.

5 The exact same trick applies to subtraction.

$$2.8 - 1.06 = 2.80 - 1.06 = 1.74$$

Try it: Calculate $4.77 - 2.3$.

In practice, most decimal calculations are done by calculator or computer. In fact, certain disciplines (such as physics and chemistry) use the number of decimals as an indication of the quality of a measurement, so that 4.21 is not the same measurement as 4.21000. So we will not be spending a lot of time performing large decimal calculations by hand. Instead, this section should be interpreted as developing a conceptual basis, not a computational basis, for decimals.

21.1 Decimal Addition and Subtraction - Worksheet 1

1 Write the number 34.72 in expanded form and label each of the parts with the corresponding unit.

2 Write the number 20.709 in expanded form and label each of the parts with the corresponding unit.

Traditionally, expanded form does not include the zeros, but it doesn't hurt anything to include them.

3 Write the number 0.73 as a fraction.

4 Write the number 0.029 as a fraction.

5 Write the number 1.05 as an improper fraction. Explain why you think your answer is correct.

Mixed numbers and addition are two ideas you can use to help explain your answer.

21.2 Decimal Addition and Subtraction - Worksheet 2

1 Calculate $2.3 + 3.08$.

2 Calculate $4.2 + 1.03$.

3 Calculate $4.6 - 2.22$.

4 Calculate $3.6 - 1.38$.

5 A student calculates $2.5 + 1.04$ and gets 3.09 as the result. How would you explain the error to them?

Just telling someone the correct way of doing a calculation does not *explain the error* to them.

21.3 Decimal Addition and Subtraction - Worksheet 3

1 Calculate $2.03 + 1.98$.

2 Calculate $1.013 + 3.92$.

3 Calculate $11.11 + 1.111$.

4 Calculate $6.1 - 3.28$.

5 Calculate $4.28 - 1.005$.

6 Calculate $5.006 - 3.09$.

21.4 Decimal Addition and Subtraction - Worksheet 4

1 The word *decimal* comes from the Latin root for ten. This is because numbers can be written in terms of powers of ten. This gives us a different way of writing the expanded form of a number:

$$\begin{aligned} 238 &= 200 + 30 + 8 \\ &= 2 \cdot 100 + 3 \cdot 10 + 8 \cdot 1 \\ &= 2 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0 \end{aligned}$$

With this framework in mind, write the number 21.84 using expanded form and showing the powers of ten.

Other examples:

- Decade: ten years
- Decathlon: ten events

You don't necessarily need to show all the steps, but you may find it helpful.

2 Write the number 107.509 using expanded form and showing the powers of ten.

3 Write the number 3100000000 using expanded form and showing the power of ten.

4 Write the number 0.000000079 using expanded form and showing the power of ten.

5 Describe some of the challenges that you faced in the last two calculations.

Scientific notation is a way of writing numbers that avoids these difficulties.

21.5 Decimal Addition and Subtraction - Worksheet 5

1 There are certain decimals that come up often enough that it is useful to be able to convert between decimals and fractions. Convert each of the decimals below into fractions, and then completely reduce them. Put the reduced fraction into the charts.

Try to organize your work so that you don't just have fractions scattered all over the place.

Decimal	0.2	0.4	0.6	0.8	0.25	0.5	0.75
Fraction							

2 There are fractions that lead to decimals that repeat a pattern forever. We can indicate repeating decimals in two different ways. One notation writes out enough of the number so that the pattern is "obvious" and then uses an ellipsis to indicate that the pattern continues. The other way uses a bar over the part of the number that repeats. Using a calculator or long division, complete the chart of values.

Fraction	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{6}$
Ellipsis Notation	$0.16666\ldots$			
Bar Notation	$0.1\overline{6}$			

3 Some decimals have interesting patterns that can be explored. One of the more surprising examples of this happens with fractions with 7 in the denominator.

If you want to play with another one, try playing with fractions with 9, 99, and 999 in the denominator.

Fraction	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$
Decimal	$0.\overline{142857}$	$0.\overline{285714}$	$0.\overline{428571}$	$0.\overline{571428}$	$0.\overline{714285}$	$0.\overline{857142}$

What pattern do you observe in these decimals?

21.6 Deliberate Practice: Adding and Subtracting Decimals

Focus on these skills:

- Write the original expression.
- Focus your attention on the position of the digits relative to the decimal point.
- Present your work legibly.

Instructions: Perform the indicated calculation.

1. $3.14 + 1.08$
2. $2.4 + 3.81$
3. $4.07 - 2.78$
4. $3.3 - 1.07$
5. $5.32 + 2.079$
6. $1.6 + 2.504$
7. $2.054 - 1.23$
8. $3.13 - 1.307$
9. $12.08 + 3.127$
10. $11.37 - 2.037$

21.7 Closing Ideas

For the overwhelming majority of real life situations (especially work situations) that you can imagine, if there are any parts of a whole involved, it will probably be done with decimals. And in most of those real life situations, if you need to do a calculation, you would do it with a calculator. So why do we need to learn how to do these basic calculations, and what is the value of relating things back to fractions?

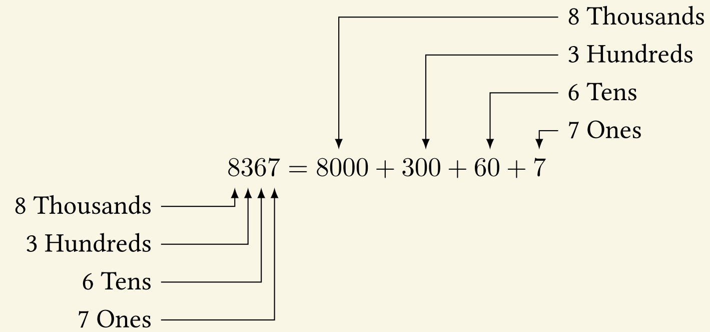
Remember that the ability to calculate a number and the ability to understand that number are two separate skills. If you type in some calculations and get 5.07 and 5.1 as the results, you may still need to decide which of those is greater. You might be surprised (or maybe not) that a fair number of adults will get that comparison wrong. Part of this is simply on the level of numerical and computational literacy, which is to help you correctly understand and use information in the real world.

The importance of being able to work with fractions is that there's a lot of information that is better communicated and more accurately communicated using fractions than decimals. For example, there are situations where it's easier to think about getting 8 items for $3 than it is to think about getting each individual item for approximately $0.38 each. And that ratio is much more useful as the unreduced fraction $\frac{\$3}{8\text{ items}}$ if you need to buy in increments of 8 (like hot dog buns).

The value is not only in having two different ways of thinking about it, but also being able to relate the two together. Sometimes fractions are the better tool and sometimes decimals are the better tool. And so it is beneficial to be able to take the information you have and apply the right tool to solve the problem instead of forcing yourself to use the wrong tool because you don't know how to use the other one. There's a well-known saying to this effect: "If all you have is a hammer, everything looks like a nail."

21.8 Solutions to the "Try It" Examples

1



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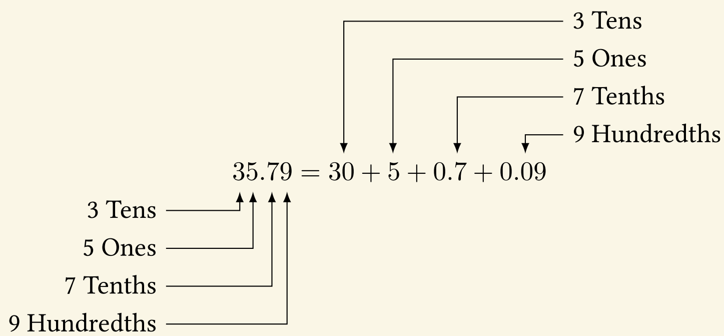


3

$$0.086 = \frac{86}{1000} = \frac{86}{10^3}$$

4

$$2.5 + 1.22 = 2.50 + 1.22 = 3.72$$

5

$$4.77 - 2.3 = 4.77 - 2.30 = 2.47$$

What Percent Do You Understand? Multiplying Decimals and Percents

Learning Objectives:

- ☐ Understand the "decimal rule" for multiplying decimals.
- ☐ Multiply by decimals.
- ☐ Convert between fractions, decimals, and percents.
- ☐ Solve basic percent problems.

There is a method that most students learn for multiplying decimals.

- Step 1: Multiply the numbers together as if there were no decimals.
- Step 2: Count the total number of decimals in the two numbers being multiplied together.
- Step 3: Place the decimal point so that the answer has the same number of decimals as the number counted in Step 2.

For example, to calculate $(1.1) \cdot (0.36)$:

- Step 1: $11 \cdot 36 = 376$.
- Step 2: The number 1.1 has one decimal and the number 0.36 has two decimals, resulting in three total decimals.
- Step 3: Write 376 with three decimal places: 0.376

The last step is very rule-minded, as it would be wrong to write 376 as 376.000 even though that technically has three decimal places. And students have to learn how to handle the special case when there are more decimal places than digits.

If you compute $(0.2) \cdot (0.4)$, the result after the first step is 8, and you don't have two digits in that number for your two decimals. It turns out you have to have leading zeros: 0.08.

But with all of this, we come back to the question that we've hit many times in this book: Why is this the rule? Why do decimals have this strange place-counting rule that we have to learn in order to do decimal multiplication correctly? It turns out that the answer comes from fraction multiplication.

1 Let's look at the example calculation again, but this time work through the framework of fractions. We will first rewrite the decimals as fractions and then multiply straight across:

$$(1.1) \cdot (0.36) = \frac{11}{10} \cdot \frac{36}{100} = \frac{11 \cdot 36}{10 \cdot 100} = \frac{396}{1000} = 0.396$$

Recall that the number of zeros of the power of 10 in the denominator corresponds to the number of decimal places.

Notice that we can immediately see that the numerator is the product in Step 1 of the process. Steps 2 and 3 in the process come from the denominator. It's basically just tracking the power of 10 that comes out in the product. And that's all there is to the rule.

Try it: Calculate $(2.4) \cdot (0.03)$ using fraction multiplication.

One of the main applications of decimals comes from percents. Most people know the rule that to convert a number to a percent, you move the decimal two places to the left. For example, 50% is just 0.50 (which is often written just as 0.5), and 237% would be written as 2.37. Again, we want to transition from this simply being a rule of percents and turn it into a concept relating decimals and percents.

2

The word "percent" can be interpreted literally as "per hundred." So when we write 75%, we're saying 75 parts per hundred. Notice that this reflects the language of parts of a whole, which leads us to think about this as a fraction. And once we write this as a fraction, we can see why it's the same as just moving the decimal two places to the left.

$$75\% = \frac{75}{100} = 0.75$$

Also notice that $\frac{75}{100}$ can be reduced to $\frac{3}{4}$. You may want to revisit Worksheet 5 of the "Decimal Addition and Subtraction" section for help with some of these fraction to decimal conversions.

In general, you will want to be able to move fluidly between all three of the notations (decimals, fractions, and percents).

We see "cent" to represent 100 in words like century (100 years) and in the fact that there are 100 cents in a dollar.

Try it: Complete the chart. Reduce the fractions where possible.

Decimal	0.37	0.25				
Percent			23%	20%		
Fraction					$\frac{14}{100}$	$\frac{3}{5}$

3

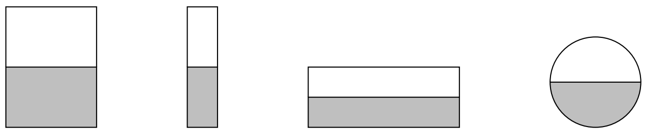
When all the numbers are written with just two decimal places, it makes the conversion to percents and fractions simple. And while there is nothing different when working with other numbers of decimals, they are prone to more errors. Just be careful and use your multiple ways of thinking about these numbers be a guide to help you determine if you have made an error.

Try it: Complete the chart. Reduce the fractions where possible.

Decimal	0.008	1.88				
Percent			123%	0.003%		
Fraction					$\frac{177}{1000}$	$\frac{13}{10}$

We have talked about the literal meaning of a percent, but we haven't talked about its practical significance. Why do we even care about percents? A percent gives us a way of thinking about values relative to other values. For example, \$1000 is a lot of money when you're buying a dinner, but it's a small amount of money when you're buying a home. So a percent gives us a relative framework to help us understand the size of something. In fact, it is precisely the size of the part relative to the whole.

Here are some visual examples of 50%. The total area of the figure doesn't matter. We are just thinking about the part of it relative to the whole amount. Percents are a way of doing that in a uniform manner.



50% 50% 50% 50%

When we talk about percentages, we often talk about a percent "of" something else, and the something else represents the whole. But sometimes we have to use context and reading comprehension to determine what the part is.

- 10% of students don't understand percents. (The whole is "all students" and the part is "the students that don't understand percents.")
- 70% of the budget went to salaries. (The whole is "the budget" and the part is "salaries.")
- Save 20% off of the regular price! (The whole is "the regular price" and the part is "the discount.")

Algebraically, we might write the relationship as

$$(\text{the percent}) = \frac{(\text{the part})}{(\text{the whole})}$$

which could be written equivalently as

$$(\text{the part}) = (\text{the percent}) \cdot (\text{the whole}).$$

This second version can be read as "the part is the percent of the whole." And this phrasing is a helpful reminder of the meaning of percents.

A less fortunate phrasing that students learn is "is over of." This language may help students to set up calculations when problems are worded in a specific way, but it gives very little insight into the actual meaning of percents. In real life, people don't go around asking "What is 20% of 45?". It usually comes in a less structured form like the following: "The bill was $45. How much should we tip?" Although we can translate the second form into the first after we understand percents, that translation step is where the actual understanding of percents is found, and the calculation is just the execution of the idea.

4 The primary exercise for thinking about percents is the practice of identifying the whole, the part, and the percent. While this is a useful mental framework, it's also important to recognize that not every single problem will fit into this mold, and that more complex problems require more complex problem-solving skills. For the most part, we will keep things simple.

The problems you will be given will be short snippets that contain information that can be translated into a percent calculation. Your task will be to identify the part, the whole, and the percent using a complete sentence. One of these will always be an unknown quantity. Then

Complete sentences matter here. The words overlap and collide in a way that can make single word answers ambiguous. Sentences force you to think it through completely.

you will need to solve for that unknown quantity and then use that information to address the question. Here is a full example:

The last batch of 500 light bulbs had 50 defects. What is the percent of defective bulbs?

This is the statement of the problem.

- The part: The number of defective bulbs is 50.
- The whole: The total number of bulbs in the batch is 500.
- The percent: The percent of defective bulbs is unknown.

This is the identification step. Notice that these are written as sentences.

$$\begin{aligned}(\text{the part}) &= (\text{the percent}) \cdot (\text{the whole})\\ 50 &= x \cdot 500\\ \frac{50}{500} &= x\\ \frac{1}{10} &= x\\ x &= 10\%\end{aligned}$$

This is the calculation.

Answer: 10% of the bulbs were defective.

This is the part that addresses the question. Again, notice that this is a complete sentence.

The amount of writing for these problems is a bit larger than usual. The reason for this is that how you write affects how you think, and we are focused on developing your thinking more than just driving you through some more algebraic manipulations. You should be able to perform the calculations in this section without a calculator.

Try it: We bought 150 balloons for the party and we've blown up 80% of them. How many balloons have we filled?

Use the structure provided above.

22.1 Multiplying Decimals and Percents - Worksheet 1

1 Calculate $(0.03) \cdot (1.5)$ using fraction multiplication.

2 Calculate $(0.42) \cdot (0.2)$ using fraction multiplication.

3 Complete the chart.

Decimal	0.1	0.2	0.25	0.3	0.4	0.5
Percent						
Fraction						

4 Complete the chart. Reduce the fractions where possible.

Decimal						
Percent	60%	70%	75%	80%	90%	100%
Fraction						

22.2 Multiplying Decimals and Percents - Worksheet 2

1 Calculate $(0.05) \cdot 30$ using fraction multiplication.

2 Calculate $(0.008) \cdot 2000$ using fraction multiplication.

3 Calculate $20\% \cdot 50$.

This is the calculation for "20% of 50."

4 Complete the chart. Reduce the fractions where possible.

Decimal	0.413	0.45				
Percent			0.7%	125%		
Fraction					$\frac{29}{100}$	$\frac{7}{5}$

22.3 Multiplying Decimals and Percents - Worksheet 3

1 What is 25% of 80?

The "classic" percentage problems are all worded like this, which leads to the "is over of" understanding of percents. Try to retrain yourself to think about this as parts of a whole.

2 30% of what number is 45?

3 What percent of 20 is 40?

Be careful!

4 The recipe states that the amount of water that is needed is equal to 70% of the weight of the flour used. If 2500 grams of flour are used in the recipe, what is the weight of water that is required for proper hydration?

- The part:
- The whole:
- The percent:

Answer:

This is known as baker's percentages, and professional bakers really do use this.

Write in complete sentences.

22.4 Multiplying Decimals and Percents - Worksheet 4

1 Calculate $10\% \cdot 5$.

2 The marketing department has $\$100,000$ to spend on this project, which is 20% of annual budget. How much money did they have budgeted for the entire year?

- The part:
- The whole:
- The percent:

Answer:

3 The car has a 12 gallon tank. When we went to the gas station, we filled it with 9 gallons of gas and now the tank is full. What percent of the tank's capacity was the gas level at before we went to the gas station?

Think carefully about the problem! The answer is not 75%.

- The part:
- The whole:
- The percent:

Answer:

22.5 Multiplying Decimals and Percents - Worksheet 5

1 A shirt was marked on sale at 20% off the regular price. The cost of the shirt at the register was \$12. What is the regular price of the shirt?

- The part:
- The whole:
- The percent:

Answer:

This is another tricky problem.

2 In the previous problem, a common error for students to make is that they compute 20% of \$12 (which is \$2.40) and then add that amount to \$12 to get \$14.40. Give two different explanations for why this approach is not correct.

3 A 20% tip can be calculated using the following method: (1) Start with the total bill; (2) Move the decimal point one space to the left; (3) Double that new number. Use algebra to explain why this results in a 20% tip.

22.6 Deliberate Practice: Percent Calculations

Focus on these skills:

- Determine which of the part, the whole, and the percent you are given in the initial statement.
- Set up the equation and solve it.
- Present your work legibly.

Instructions: Answer the question.

1. What is 20% of 70?
2. 50% of what number is 15?
3. 40 is 10% of what number?
4. What percent of 60 is 18?
5. 5% of 60 is what?
6. What is 30% of 27?
7. 8 is 25% of what number?
8. 40% of 32 is what number?
9. 25% of what number is 12?
10. What number is 75% of 52?

22.7 Closing Ideas

As mentioned before, most decimal multiplication is performed by calculators or computers. But computers are usually not able to perform percent calculations without a person correctly identifying the part, the whole, and the percent, and then determining what calculations are required to solve the problem. With the goal of mathematical thinking in mind, the problems in this section were set up so that calculators would not be needed.

But if you were run into a situation where you would need a calculator, as long you have the correct mathematical thinking then all you need to do is replace the mental calculation with a calculator calculation. There is no real loss (from the perspective of logic) in trading that out. Here is the light bulb problem again, but with slightly more realistic numbers:

The last batch of 1500 light bulbs had 37 defects. What is the percent of defective bulbs?

- The part: The number of defective bulbs is 37.
- The whole: The total number of bulbs in the batch is 1500.
- The percent: The percent of defective bulbs is unknown.

$$\begin{aligned}
(\text{the part}) &= (\text{the percent}) \cdot (\text{the whole}) \\
37 &= x \cdot 1500 \\
\frac{37}{1500} &= x \\
x &\approx 0.02467 \\
x &\approx 2.467\%
\end{aligned}$$

This number came from a calculator

Answer: About 2.5% of the bulbs were defective.

Notice that the overall process is unchanged, and it's just a matter of using different numbers. This is very similar to the process of learning to treat variables like numbers in other calculations, such as reducing fractions. In fact, the particular area of mathematical thinking, which is sometimes called *algebraic reasoning*, is the whole idea that you can generalize the methods and ideas of simple examples so that they can be applied in more complex situations. You got a hint of this type of reasoning in the last few problems where you had to do a mental manipulation before putting the numbers into the parts of a whole framework. And that is the skill that you should be aiming to develop in your college level courses. You want to be more than a calculator.

22.8 Going Deeper: Proportional Reasoning

In this section, we focused on the standard type of percent problem and some minor variants of it. With some of the more complicated problems, you had to do a bit more thinking to correctly identify the components of the basic percent relationship. This way of thinking about percents is a specific example of a broader framework known as *ratios*. A ratio is a general type of mathematical relationship where two (or more) quantities are held in a constant proportion with each other.

We will see ratios again when we look at unit conversions.

For example, if you are buying packages of hot dog buns where each package contains 8 buns, then there is a constant ratio between the number of packages you purchase and the total number of buns you have. We can set this up using the following word equation:

$$\text{(the number of buns)} = 8 \cdot \text{(the number of packages)}.$$

When you compare this to the percent equation, you'll see that it has a similar structure, though the name of the components are different.

We can set this up a little more generally:

$$\text{(the number of item } Y) = \text{(the ratio of item } Y \text{ to item } X) \cdot \text{(the number of item } X)$$

We usually prefer to use symbols instead of words because it takes up a lot less space, so if we let y represent the number of item Y, x represent the number of item x, and let k be the ratio of item Y to item X, then this becomes

$$y = kx.$$

This type of relationship is one of the standard models that we use for talking about how two quantities are related to each other. We call this a linear (or direct) relationship between the variables. In many cases, it's more useful to think of the equation in the form

$$k = \frac{y}{x}$$

which more directly shows us that k is the ratio of the number of item y to the number of item x.

We can set up this relationship between any two collections of objects. In example above, it was the ratio of hot dog buns to the number of packages. When we're talking about percents, it's the ratio of "the part" (the number of a specific type of object) to "the whole" (the total number of objects in the collection). We also saw this ratio when looking at slopes of lines (the amount of "rise" to the amount of "run"). When we think about speed, we think about the ratio of how far something travels for a given amount of time, which is why speed has units such as "miles per hour." This shows that idea of a ratio is fundamental to algebraic reasoning and is useful in many applications.

It is often useful to think about the distinction between a ratio and a proportion. A ratio is the relationship between two quantities. A proportion is when we say that two ratios are the same. For example, if you went to the store to buy two packages of hot dog buns and your friend went to the store to buy two packages of hot dots, you may end up with different numbers of buns and dogs even though you bought the same number of packages. The reason for this is that the buns and dogs may have different numbers of objects per package. They are not proportional to each other.

Proportional reasoning can be difficult for many people because the the importance of one quantity is measured relative to the size of another. For example, losing $1000 can be very detrimental to a household's income, but a multi-billion dollar company would hardly be bothered by the loss. And on the other side of things, the difference between the price of milk being $2.99 per gallon or $3.09 per gallon is negligible for a household's budget, but this can be a large additional expense for a company that needs to buy millions of gallons of milk.

This idea can be looped back around to percents by thinking about *percent change*. The idea of a percent change is that we're looking at the ratio of the amount of change to the quantity relative to the quantity itself. This helps us to think about how much of an impact something has on the overall situation. For example, a pay raise of $1 per hour means a lot to someone who is earning $10 per hour, but it means very little to someone earning $100 per hour, even though the raise is the same size. The difference is that it's a larger percent increase in wages to the person earning less money (a 10% pay raise compared to a 1% pay raise).

An important note about percent change is that it can lead to error if you're not careful. An example of this can be seen if we think about something doubling. For example, let's say that the person making $10 per hour finds a different job and doubles their wages to $20 per hour. What is the percent change of their wages? Some people immediately latch on to the number 2 (because the wages doubled) and convert 2 into a percent to get a percent change of 200%. But this is wrong. We have to go back and think about what the definition of a percent increase is. We have to look at the total amount of change in the wages, which is $10 more per hour, and then use the base pay as the denominator of the ratio, which is also $10. This means that there was a 100% increase in their wages.

This is not intuitive for many people. It's a specific type of thinking that requires time and practice in order to become proficient. As you continue to take quantitative classes, especially science and social science courses, you will see proportional reasoning start to seep into the basic language you use to describe the world around you, and the more prepared you are, the more you will get out of those other classes.

22.9 Solutions to the "Try It" Examples

1

$$(2.4)\cdot(0.03) = \frac{24}{10}\cdot\frac{3}{100} = \frac{24\cdot 3}{10\cdot 100} = \frac{72}{1000} = 0.072$$

2

Decimal	0.37	0.25	0.23	0.2	0.14	0.6
Percent	37%	25%	23%	20%	14%	60%
Fraction	$\frac{37}{100}$	$\frac{1}{4}$	$\frac{23}{100}$	$\frac{1}{5}$	$\frac{14}{100}$	$\frac{3}{5}$

3

Decimal	0.008	1.88	1.23	0.00003	0.177	1.3
Percent	0.8%	188%	123%	0.003%	17.7%	130%
Fraction	$\frac{1}{125}$	$\frac{47}{25}$	$\frac{123}{100}$	$\frac{3}{100000}$	$\frac{177}{1000}$	$\frac{13}{10}$

4

- The part: The number of filled balloons is unknown.
- The whole: The total number of balloons is 150.
- The percent: 80% of the balloons have been blown up.

$$\begin{aligned}
(\text{the part}) &= (\text{the percent})\cdot(\text{the whole}) \\
x &= 80\%\cdot 150 \\
&= \frac{80}{100}\cdot 150 \\
&= \frac{4}{5}\cdot 150 \\
&= \frac{4}{\cancel{5}}\cdot 30\cdot \cancel{5} \\
&= 120
\end{aligned}$$

This is the calculation.

Answer: 120 of the balloons were filled.

Another Pause for Reflection

Congratulations! You have just completed the first two "branches" of the course.

In the first branch, we looked at some of the ideas of the coordinate plane, which will be important for anything you do in the future that involves graphs. We also looked at the equations of lines, which are important for both analytical mathematical calculations as well as applied mathematical calculations. It turns out that lines are one of the core conceptual models we use in the development of calculus (approximating curves with lines) and lines are one of the core applied models we use in applications (including in the development of artificial intelligence).

In the second branch, we broadened our numerical experiences to include fractions and decimals. Fractions represent a division concept and can be interpreted using the concept of parts of a whole. We saw that addition and subtraction of fractions require us to work with a common denominator, and that the common denominator is simply a way of allowing us to work with pieces that are all the same size, rather than pieces that are of different sizes.

With that framework in mind, we saw that fraction multiplication is just a bookkeeping exercise, where multiplying the numerators is tracking the number of wedges and multiplying the denominators is tracking the size of the wedges. And this idea is what makes "multiply straight across" a logical and non-arbitrary computational method. Next, we explored the relationship between multiplication and division, and focused on the idea that these operations are inverses of each other. We also developed more computational methods for handling increasingly complex fraction division calculations.

Finally, we looked at decimals as yet another representation of parts of a whole. By thinking about common denominators, we saw that addition and subtraction was easiest to perform if we simply viewed the numbers as having the same number of digits after the decimal point. We also saw that moving the decimal around in multiplication calculations is the exact same concept we used in fraction multiplication, but presented in a different way.

At this point, we're going to take another pause in order to let you practice your metacognitive skills by thinking about what you've learned in order to continue to better understand our the development of your own thinking.

In this portion of the course, we have covered the following topics:

- Lines and the Coordinate Plane
- Slope-Intercept Form
- Solving Systems of Equations by Substitution
- Solving Systems of Equations by Elimination
- Fraction Basics
- Fraction Addition and Subtraction
- Fraction Multiplication
- Fraction Division

- Decimal Addition and Subtraction
- Multiplying Decimals and Percents

Questions About the Content

1 Were there any topics that you had seen before, but you understand better as a result of working through it again?

2 Were there any ideas that you had never seen before?

3 Based on your experience, which of these ideas seems the most important to understand well?

4 Did any part of the presentation make you curious about math in a way that went beyond the material? Are there questions or ideas that you would like to explore?

Examples:

- What are the shapes of other graphs besides lines?
- How do you do calculations with mixed numbers?
- Why didn't we look at decimal division?

Questions About You

1 How has your mathematical writing continued to evolve from the previous pause for reflection? Do you find yourself thinking in different ways?

2 Did you have any "Aha!" moments where you had an insight into something that you had not noticed before?

3 What is the biggest mathematical connection that you made?

4 How is your mathematical confidence coming out of these sections? Is it going higher, lower, or staying at about the same as it was?

Rethinking Arithmetic: Visualizing Numbers

Learning Objectives:

- ☐ Understand the representation of positive and negative numbers on a number line.
- ☐ Properly use the symbols $>$ and $<$ to write mathematical relationships between numbers.
- ☐ Interpret place values using base-10 blocks.

An important aspect of mathematical thinking is the ability to represent ideas in several different ways. Even something as simple as a number can have multiple representations, and those different representations have different applications. And it may seem simple, but even just representing numbers as a collection of dots or squares is a technique used in higher levels of mathematics to help discover patterns and prove mathematical relationships.

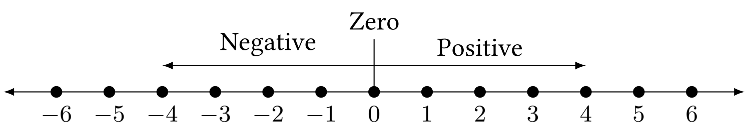
We've already used two different representations of numbers in this book. The first is the number line, and the second is the place value system. We're going to spend some time exploring these ideas a little more closely to deepen our mathematical thinking.

The number line is the ordering of numbers in a straight line based on their values. Traditionally, this line is drawn horizontally, though there is no reason that this has to be the case. In fact, the coordinate plane is actually two number lines drawn together where one of them is drawn vertically.

Horizontal lines are convenient because we read left-to-right, which creates a forward/backward intuition that helps.

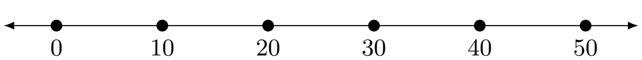
We can think of the number of 0 as being at the middle of the number line, with positive numbers to the right of it and negative numbers to the left of it.

Does it even make sense to talk about the "middle" of an infinitely long object? Maybe not, but just a basic intuition is good enough for this.

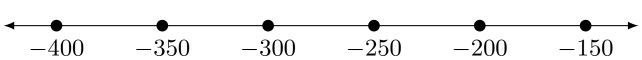


We can visualize any part of the real line that we want using any sense of scale that we want. It does not always need to include zero. If the actual location is important, then you should try to use an equal spacing. But sometimes all we need is a symbolic representation of the locations. Regardless, it is very important that we always keep the order the same, especially with negative numbers. For a portion of the line that only has negative numbers, remember that going to the left makes the numbers more negative. Here are some examples:

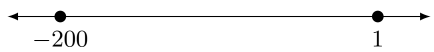
- A number line with zero and positive numbers using an increment of 10:



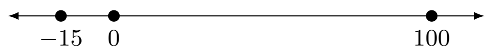
- A number line with negative numbers using an increment of 50:



- A representation of the relative locations of the numbers -200 and 1.



- An approximation of the locations of the numbers -15, 0, and 100.



As we move away from 0, the numbers get bigger in size. The phrase "bigger in size" is important, because the words "bigger" and "smaller" on their own create confusion with negative numbers. The size of a number is often called its absolute value (or magnitude). Intuitively, $-1\,000\,000$ is a "big" number by size, but it can also represent a large deficit instead of a large quantity. To avoid that confusion, we use the phrases "greater than" and "less than" when comparing numbers. These words take away the possibility of misinterpreting the comparison of two numbers.

You would prefer your bank account to be \$10 instead of $-\$1\,000\,000$ even though the latter is a "bigger" number.

Definition 24.1. For any two numbers a and b, we say that a *is greater than* b (written symbolically as $a > b$) if a is to the right of b on the number line. We say that a *is less than* b (written symbolically as $a < b$) if a is to the left of b on the number line.

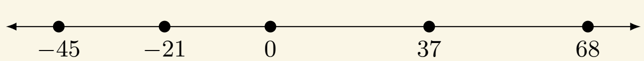
Using the number line to visualize the locations of numbers is a natural approach to comparing numbers. Some students learn a slightly complex set of rules for comparing numbers:

Don't try to learn these rules. They're not good for you.

- If both numbers are positive, then the bigger number is greater than the smaller number.
- If one number is positive and the other is negative, then the positive number is greater than the negative number.
- If both numbers are negative, then the bigger number is less than the smaller number.

While this is accurate, it ends up causing confusion because it turns it into a practice of rule-following rather than developing an understanding.

1 Ordering numbers is a skill that simply requires some practice. The best intuition comes from starting at 0 and thinking about the number of steps in which direction is required to reach a value. For example, starting from 0, to get to the number -45 you would have to move to the left, and on the way you'll pass -21. And to reach the number 37, you would start from 0 and move to the right, and when you get there, you won't have yet passed 68. This could be represented on a number line.



Try it: Put the numbers 36, 11, -58, -3, and 132 in order on a number line.

2 Once you are comfortable with locating numbers on the number line, then comparisons thinking about "to the left of" (for "less than") and "to the right of" (for "greater than") are straightforward. Based on the number line diagram above, we can immediately check the fol-

lowing comparisons:

$$-45 < 37 \qquad 68 > 0 \qquad -21 > -45 \qquad -21 < 68$$

Check these!

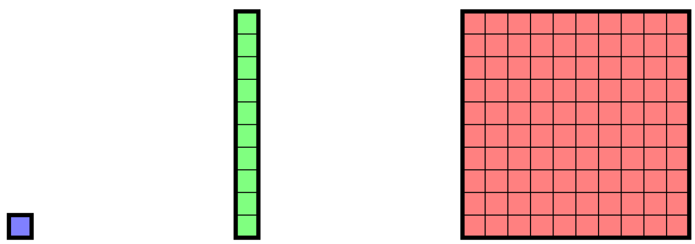
Try it: Write all 6 mathematical sentences that compare the numbers -14, 10, and 31.

The sentences $5 > 3$ and $3 < 5$ are considered to be different sentences even though they express the same fact.

The second representation of numbers that we've used is the place value system. This is the way that you're already familiar with writing numbers, and we discussed it briefly when we looked at decimals. But there is another way of looking at these numbers that have an important generalization to other mathematical ideas.

In early elementary school, a common manipulative that's used to help teach children numbers are known as *base*-10 *blocks*. These are basically just plastic or wooden pieces that come in three different shapes.

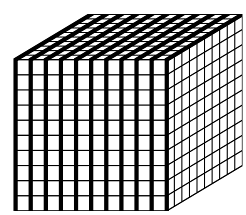
Some people use different names for these shapes. The names don't actually matter.

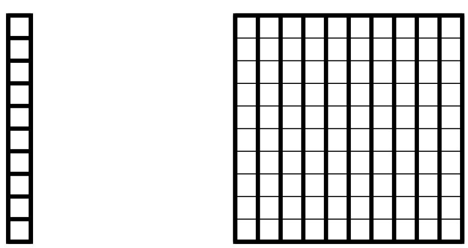


"Unit cube" "Tens rod" "Hundreds flat"

Furthermore, we have containers that the various pieces fit into. We have a tray that fits ten units and a tray that fits ten rods. We will represent these by uncolored boxes, and as pieces are put in, they will be colored in.

You can also have a large cube to line 10 up flats side by side, but I don't think I've ever actually seen one of those.



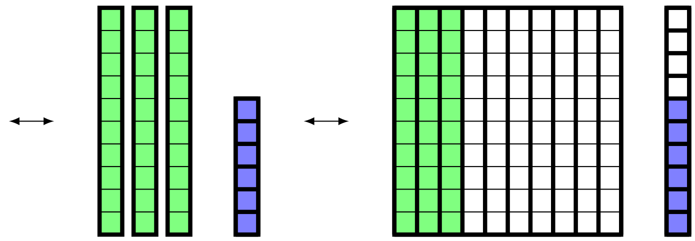


Unit cube tray Tens rod tray

Notice that an empty unit cube tray looks a lot like a tens rod, and that a tens rod tray looks like a hundreds flat. This is because in practice, students would get to exchange their full tray of unit cubes for a rod (or the other way around), and this helps to reinforce arithmetic using the base-10 system. (We'll see this again in a little bit.)

Ten unit cubes is the same as one rod. Ten rods is the same as one flat.

If we wanted to represent a number, we could simply pick the appropriate number of each piece. Here is an example:

36 

You do not need to draw the trays, though it's a good concept to have in mind for the future.

3 You should be able to go back and forth between a number and combinations of base-10 blocks in order to represent any value.

Try it: Represent the number 52 using base-10 blocks.

In the next couple sections, we will see how these base-10 blocks can be used to help explain some of the common arithmetic algorithms that are used.

24.1 Visualizing Numbers - Worksheet 1

1 Draw a number line from -10 to 10 using increments of 1.

2 Write all 6 mathematical sentences that compare the numbers -4, -7, and 5.

3 Draw a number line and give the approximate locations of the numbers -45, 22, and 65.

4 Write all 6 mathematical sentences that compare the numbers -45, 22, and 65.

5 Give a representation of the relative locations of the numbers 23 and 32 and write 2 mathematical sentences comparing them.

6 Give a representation of the relative locations of the numbers -23 and -32 and write 2 mathematical sentences comparing them.

Remember to think about starting from 0 and moving to these numbers. You should have something different from the previous problem.

24.2 Visualizing Numbers - Worksheet 2

1 Draw a number line from -100 to 100 using increments of 10.

2 Write all 6 mathematical sentences that compare the numbers -60, 0, and 80.

3 Draw a number line from -800 to 600 using increments of 25.

4 Write all 6 mathematical sentences that compare the numbers -750, -625, and -675.

5 Represent the number 34 using base-10 blocks.

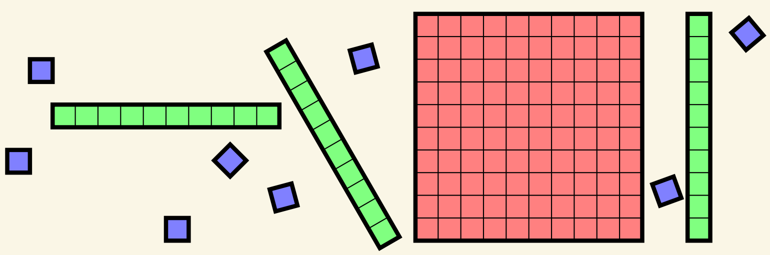
Be sure to draw all 10 squares on the rod. That is an important concept for these diagrams.

24.3 Visualizing Numbers - Worksheet 3

1 Give a representation of the relative locations of the numbers -47 and -83 and write 2 mathematical sentences comparing them.

2 Represent the number 238 using base-10 blocks.

3 What number is represented by the following base-10 blocks?



4 Explain why the above arrangement of base-10 blocks is not ideal. Then give a better arrangement and explain why it's better.

Give at least two reasons your arrangement is better.

24.4 Visualizing Numbers - Worksheet 4

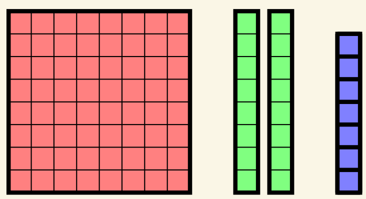
1

Although we are most familiar with base-10 number, this is not the only system of numbers that is used. Computers have three other number systems that it uses: binary (base-2), octal (base-8), and hexadecimal (base-16). We are going to explore those bases to understand how they work.

The primary difference is that the size of rods and trays are different. When working in base-10, it takes 10 pieces to go up to the next shape. In base-8, it only takes 8. Here is the visual representation of the number 127_8.

The subscript number tells us what base we're in. If there is no subscript, then the number is in base-10.

Determine what this number is in base-10 and explain your logic.



2

Converting numbers from base-10 to base-8 is a bit more complicated. Try to imagine that you have a bunch of loose blocks that you're filling into different trays that are built around the number 8 instead of the number 10. Work from the largest trays and work your way down.

Convert 89 to base-8 and explain your process in words.

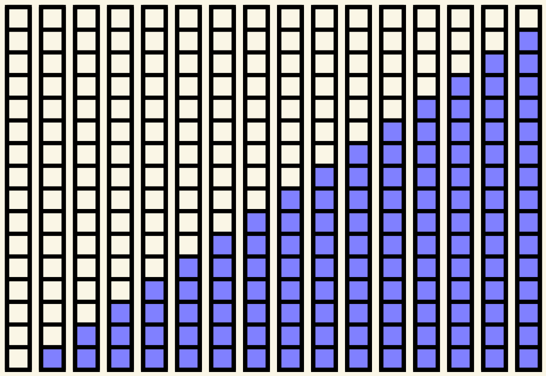
3

Using the logic that you developed, convert 14 to base-2 and explain your process in words.

24.5 Visualizing Numbers - Worksheet 5

1

Base-16 requires us to introduce more symbols into our system of digits. The following diagram represents all of the single-digit numbers in that system.



$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad A \quad B \quad C \quad D \quad E \quad F$

Based on this diagram, what do you think the base-10 representation of 10_{16} is? Explain your logic.

2

Convert AC_{16} to base-10. Explain your reasoning.

24.6 Deliberate Practice: Inequalities and the Number Line

Focus on these skills:

- Try to make the spacing as even as possible for the equal increments.
- Pick increments that make sense for the problem. Note that the numbers you plot may not always fall precisely on the marked increments.
- For each comparison, think through their relative locations ("to the left of ‘’ or "to the right of").
- Present your work legibly.

Instructions: Draw a number line with equal increments and give the approximate locations of the given numbers, then write all 6 mathematical sentences comparing them to each other.

1. $4, 9, 6$
2. $-7, 0, 5$
3. $20, -10, -60$
4. $-300, 700, 300$
5. $31, 55, 22$
6. $-27, 14, -49$
7. $39, -25, 0$
8. $-43, -11, -68$
9. $183, 832, 438$
10. $-273, 185, -487$

24.7 Closing Ideas

The ability to see the same idea from multiple perspectives creates the opportunity to apply different approaches to solving problems. In this section, we saw two different ways to represent numbers.

The number line is a purely geometric framework. It turns out that the Greek mathematicians thought of numbers this way, but in an even more strict sense. If they wanted to compare the numbers 5 and 8, then they would (essentially) say that 8 is longer than 5. But how would this work with negative numbers? As it turns out, the Greeks never really bothered with negative numbers. To them, they didn't exist because they only worked with counting numbers and lengths. Since the number line contains negative numbers, we will have an additional set of tools that the Greeks did not have when it comes to thinking about numbers and mathematical ideas.

The base-10 blocks gives us another geometric framework, but it ties more closely with our sense of how we represent numbers instead of giving us insights into the numbers themselves. It may seem obvious to us because we've worked with numbers this way our whole lives. But not every system of writing numbers uses a place value system. For example, Roman numerals are notoriously difficult for students to learn because it's built around rules that are not always intuitive and sometimes feel arbitrary.

What comes next in the following pattern?

- $1 \longrightarrow \text{I}$
- $2 \longrightarrow \text{II}$
- $3 \longrightarrow \text{III}$

Nobody would ever guess IV comes next if they didn't already know Roman numerals.

As you come to understand more mathematics, you can start to develop a more flexible mindset for looking at ideas. This can often lead to new insights and more interesting questions.

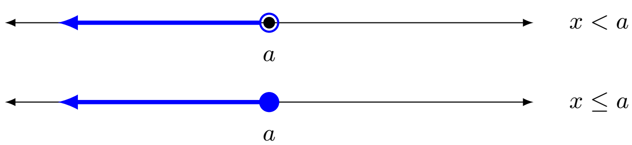
24.8 Going Deeper: Intervals on the Number Line

In this section, we introduced a couple geometric representation of the numbers. We are going to focus our attention on the number line picture in the context of thinking about inequalities. Suppose that we have the inequality $x > a$. Here's how we would represent that on the number line:

Inequalities were discussed in Section 4.8.

The circle around the a indicates that we do not want to include a in the interval. It represents a "hole" in the arrow at that point. If we wanted to graph $x \geq a$, then it would look like this:

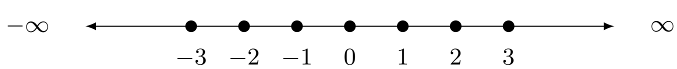
We could draw similar diagrams for $x < a$ and $x \leq a$:



The arrows on the end of the thickened lines indicates that it extends forever in the indicated direction. This introduces the concept of infinity, which is denoted ∞. We are going to have to leave this at an intuitive level, as infinity turns out to be an incredibly complicated and nuanced topic. The key fact is that infinity is *not* a number. It is not a part of the number line. For our purposes, it represents the idea of continuing along in the same direction indefinitely. We also have an infinity in both directions, where ∞ (sometimes written $+\infty$ for emphasis) is off to the right and $-\infty$ is off to the left. The diagram below is an attempt to convey this idea:

Look up the "Hilbert Hotel" if you're interested in exploring the nature of infinity.

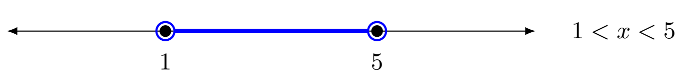
Notice that neither ∞ or $-\infty$ have dots associated with them, and that they are not attached to the number line. Both of these features emphasize that they aren't numbers.



There are times that we want to restrict our values on two sides. For example, if you want a number between 1 and 5 (not restricting yourself to integers), you're actually asking for the number to meet two conditions at the same time: (1) The number must be greater than 1; (2) The number must be less than 5. This is an example of a compound inequality.

Graphically, this is not too hard to think about, as the betweenness property is captured intuitively.

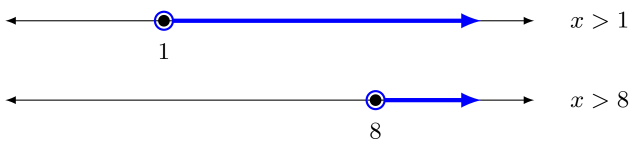
There are two types of compound inequalities. We can combine them with the word *and* to indicate that we want both conditions to be met, or we can combine them with the word *or* to indicate that we want at least one of the conditions to be met. We are going to focus on the first type here.



Notice that we wrote this as a single inequality. Even though this looks like just one inequality, it's actually shorthand for two inequalities: $1 < x$ and $x < 5$.

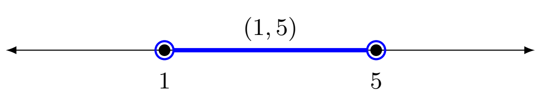
It's important that the direction of the inequality is consistent. If the inequalities get turned around, we treat it as a meaningless statement. For example, $1 < x > 8$ is *not* interpreted as $1 < x$

and $x > 8$. If we look at what that would mean, we can see that there's a bit of redundancy in that interpretation. We would be looking for values to the right of 1 that are also to the right of 8. But every number to the right of 8 is already to the right of one, so the first part doesn't really add anything but confusion. So it leads to cleaner communication to declare that such combinations are not allowed.



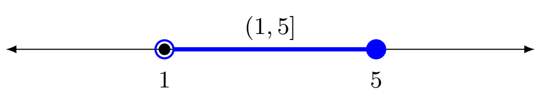
Sets of these types are known as *intervals*. They play an important role in understanding and describing functions and other mathematical objects. We have seen that we can describe intervals using diagrams and inequalities, and it turns out that there is one more method that we use, which is known as *interval notation*. The value of interval notation is that it allows us to describe an interval using symbols, but without introducing a variable. When talking about a number between 1 and 5, it's not necessarily helpful for us to arbitrarily pick a symbol to represent that quantity. This is especially true for more complicated situations where we're already working with several other variables.

Interval notation is easiest to understand through to the number line diagrams that we've been drawing. There are just a couple ideas that we need. The first idea is that we always want to think about the diagram from left to right. And this is easy to remember because your notation will be written out from left to right. The second is that we use a round bracket to exclude the endpoint and a square bracket to include it. Here is the example we were working with before.

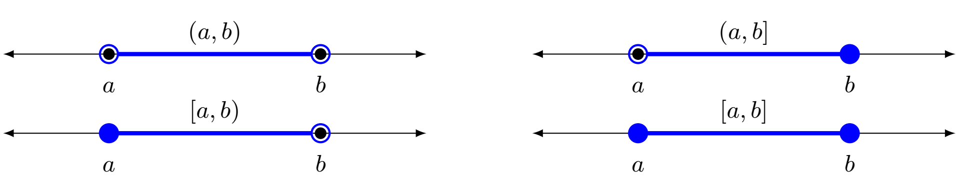


How can you tell the difference between the interval $(1, 5)$ and the point $(1, 5)$? Use context clues.

Here is the same example, except that we're including the point 5 in our set.

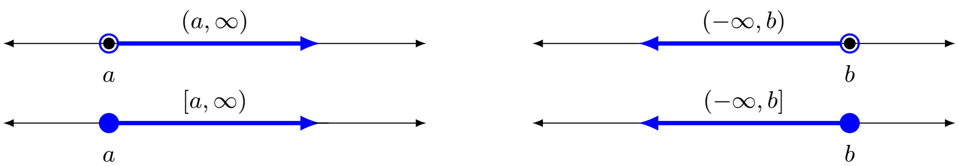


As you can see, there are four different combinations of symbols that we might have, depending on whether each of the endpoints are included or excluded. Here they are presented together:



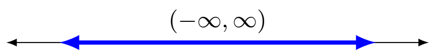
What about intervals that go off to infinity? The same ideas hold. In particular, we always use round brackets around the side with infinity because infinity is not actually included in the interval. We also need to use the appropriate sign on the infinity, depending on which side we're considering.

Remember: Infinity isn't a number!



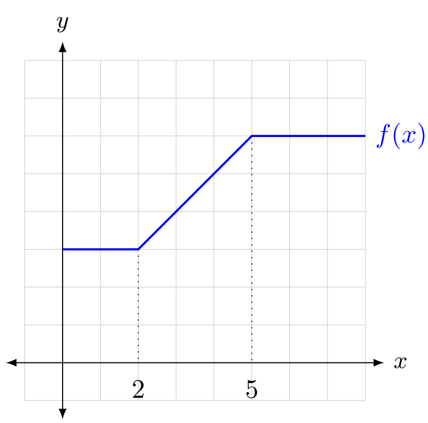
We also have one more case where both sides go to infinity. This is basically saying that we want to include every real number. Fortunately, this case follows all the same ideas as before, so the interval notation should not be a surprise.

Some people will denote this set as $\mathbb{R}$.



With these examples in mind, you should be able to represent any interval using three different representations: a number line diagram, an inequality (or compound inequality), and interval notation. You should also be able to move freely between the different forms. For example, if you're given a number line diagram, you should be able to translate that into an inequality as well as writing it in interval notation.

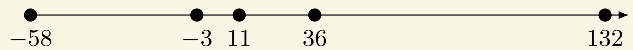
Interval notation is used very often as part of the larger language of mathematics. While the notation is easy to understand on its own, it can be difficult to understand how we use it in applications without placing it in a larger context. Those larger contexts require some ideas that go beyond what we've developed. We will use one example that involves describing the behavior of a function. It will hopefully be intuitive, but don't worry too much if it's not. Consider the following graph:



The function is increasing on the interval $(2, 5)$. In other words, the function is increasing when $2 < x < 5$.

24.9 Solutions to the "Try It" Examples

1

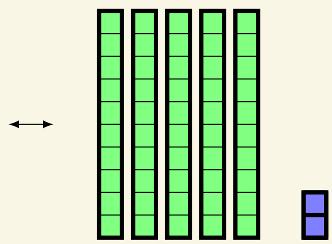


2

$-14 < 10$	$-14 < 31$	$10 < 31$
$10 > -14$	$31 > -14$	$31 > 10$

3

52



Carry the One (or Maybe Not): Addition

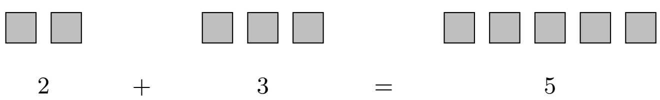
Learning Objectives:

- ☐ Understand what it means to "carry the one."
- ☐ Understand how addition can be visualized on a number line.
- ☐ Develop effective strategies for mental addition for 2-digit and 3-digit numbers.

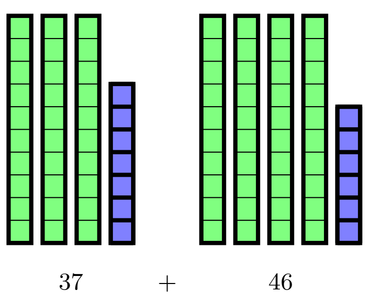
The idea of "carrying the one" is one of the many phrases that are taught when it comes to arithmetic. We are going to take an unusual approach in that we're going to talk about what it is and how it works, but then we're going to talk about why it's not that important in practice (at least these days).

It is so common that there is a trope of cartoon characters muttering "carry the one" when they're engaged in a moment of complex thinking.

Before we can talk about "carrying the one," we first need to think about what addition is. One way to think about addition is that you're starting with a certain quantity, and then we're going to combine it with another quantity and look at how much we have in total. When working with young children, the picture looks something like this:

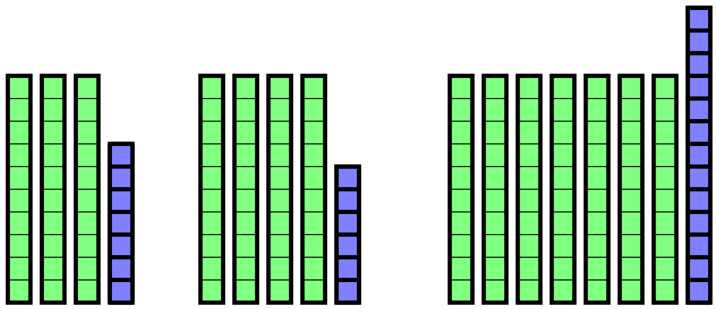


But as the numbers get larger, our methods need to get more sophisticated. First, instead of just scattered blocks as we have above, we're going to use base-10 blocks. And then we're going to have to think logically about how we organize that information. Here is a diagram for $37 + 46$:

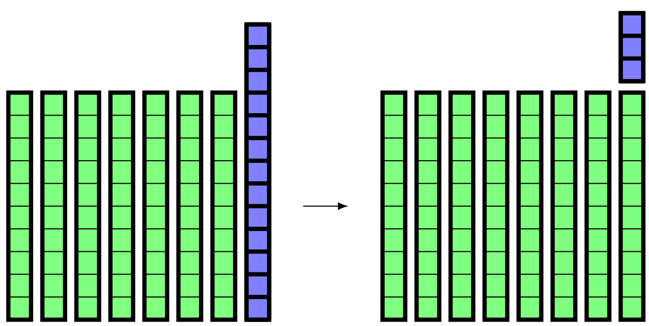


The most natural thing for us to do here is to reorganize the information so that the unit cubes are together and the tens rods are together.

This is the same concept as combining like terms.



This diagram highlights the reason that "carrying the one" is part of the process. We have run into the situation that we have "too many" unit cubes. They spill over to a number larger than what we can account for with the place value system. And so that's where we trade in 10 units for 1 rod, which is the concept that is attached to "carrying the one."



1

Try it: Draw a base-10 blocks diagram to represent $35 + 17$ and compute the result.

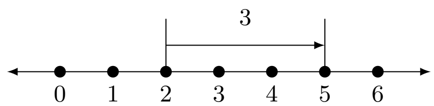
In practice, having actual manipulatives is better than drawing pictures of manipulatives. But as far as this textbook goes, this is as close as we're going to get.

The idea of "carrying the one" was developed for the purpose of pencil-and-paper arithmetic. If you needed to add a long column of numbers (and there were once good jobs out there for people who can do this quickly and accurately), then you needed a notation for the number of groups of 10 that needed to be accounted for in the next larger place value. But at this time, there is not a lot of value in this particular calculation because we can have computers do it many times faster than we can and they do it with perfect accuracy.

But this does not mean that there is zero value in humans performing arithmetic. There are times when it's handy to be able to do a 2-digit or 3-digit addition problem mentally instead of having to reach for a calculator. Some people are able to add in columns in their heads, but many find it difficult to keep track of all the different digits floating around. So we are going to work with addition on the number line as our model for thinking about mental arithmetic.

Did you ever notice that when you add in columns that you construct the number backwards? That gives a big hint as to why it feels complicated for a lot of people.

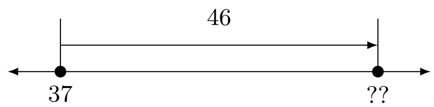
Addition on the number line is about movement. The first number represents your starting position and the second number represents how far to the right you move from that position. Here is what $2 + 3$ looks like:



$\boxed{2 + 3 = 5}$

- Step 1: Start at 2.
- Step 2: Move 3 spaces to the right.
- Step 3: End up at 5.

For small numbers, it is easy enough to just count out the steps. But for a larger calculation, counting is simply too slow. And in order to simplify the diagram, we're going to use just the part of the number line that's relevant. Here is the setup to calculate $37 + 46$:



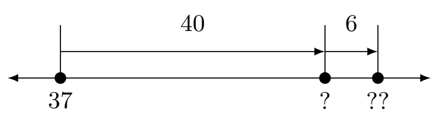
$\boxed{37 + 46 = ??}$

- Step 1: Start at 37.
- Step 2: Move 46 spaces to the right.
- Step 3: End up at ??.

Of course, the challenge is to figure out what the value of ?? is going to be.

Remember that our goal is to set ourselves up for mental arithmetic. So we are going to set up this picture in a way that can be done with simple mental calculations, rather than working with base-10 blocks or digit manipulations. The key trick is to break the motion of 46 steps to the right into two separate motions: 40 steps to the right followed by 6 more steps to the right. In the diagram below, see if you can work out the values of both ? and ?? by thinking through the picture.

This is not that different from counting out change. You start with the larger denominations first, and then work your way down to the smaller ones.



$\boxed{37 + 46 = ??}$

- Step 1: Start at 37.
- Step 2: Move 40 spaces to the right.
- Step 3: Move 6 spaces to the right.
- Step 3: End up at ??.

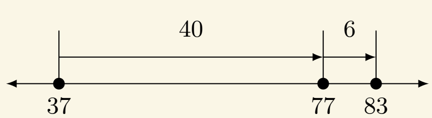
With a little bit of mental effort, you should be able to determine that ? is 77 and ?? is 83.

What's interesting about this is that you probably didn't have to think about "carrying the one" at all when going from 77 to 83. When looking at the number line, you would never make the mistake of thinking that $77 + 6$ is 73 (forgetting to "carry the one") or 713 (incorrectly placing the 1 between the ones digit and tens digit). But when we teach children to add in columns, they make these mistakes with regularity. And this highlights the difference between working with adults and working with children. Many children are still developing their basic number sense, but most adults already have it, and so we can leverage that number sense into methods that are feel far more intuitive for adults than children.

Calculate $999 + 1$. Did you "carry the one" three times to get the number 1000? Probably not. So "carrying the one" isn't actually necessary.

2

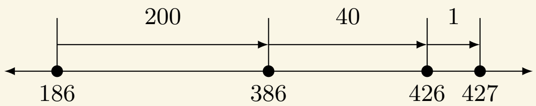
Here is the full diagram for the calculation $37 + 46$ using a number line.



$\boxed{37 + 46 = 83}$

Try it: Calculate $39 + 27$ using a number line.

3 The same idea can work for larger numbers, and it's not significantly more mentally taxing.



$\boxed{186 + 241 = 427}$

Try it: Calculate $271 + 119$ using a number line.

4 Drawing the number line is important to practice the mental organization, but in order for this to be a mental calculation, you need to be able to do it without drawing the picture. (But you might find it helpful to have the picture in your head!)

Try it: Calculate $183 + 319$ mentally.

Some people find it helpful to physically drag their finger to the right as they do each step.

This takes practice. Take your time. This is not a race.

25.1 Addition - Worksheet 1

1 Draw a base-10 blocks diagram to represent $38 + 15$ and compute the result.

2 Draw a base-10 blocks diagram to represent $195 + 219$ and compute the result.

3 Think about (do not draw) the diagram you would need to represent $48 + 37$ and compute the result from that mental picture. Did you find the visualization helpful or distracting? Explain what was helpful or distracting about the mental image for you.

The only correct answer to the "helpful or distracting" part of the question is an answer that is true to your own experience.

25.2 Addition - Worksheet 2

1 Calculate $36 + 59$ using a number line.

2 Calculate $227 + 389$ using a number line.

3 Think about (do not draw) the number line diagram you would need to calculate $23 + 38$ and compute the result from that mental picture. Do you prefer this visualization or the base-10 blocks visualization? Why?

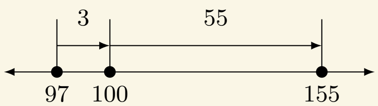
4 Practice your mental arithmetic by performing the following calculations.

$44 + 21 =$	$24 + 68 =$	$46 + 19 =$
$16 + 25 =$	$79 + 26 =$	$34 + 18 =$
$323 + 258 =$	$297 + 197 =$	$186 + 315 =$

You might be surprised how much a little bit of practice can help to develop your skill and your confidence.

25.3 Addition - Worksheet 3

1 Here is a trick for mental arithmetic when one of the numbers is close to a multiple of 10 or 100. Figure out how far you are from that "nice" number and move that many steps first. Then the remaining movement is much easier to do. Here's a visualization of $97 + 58$:



$\boxed{97 + 58 = 155}$

Draw a number line diagram using the technique above to calculate $98 + 77$.

We're actually using subtraction to facilitate addition.

2 Draw a number line diagram using the technique above to calculate $49 + 36$.

3 Practice your mental arithmetic by performing the following calculations.

$96 + 47 =$	$97 + 88 =$	$37 + 94 =$
$49 + 76 =$	$38 + 84 =$	$34 + 18 =$
$197 + 377 =$	$298 + 428 =$	$415 + 189 =$

Remember that $a + b = b + a$.

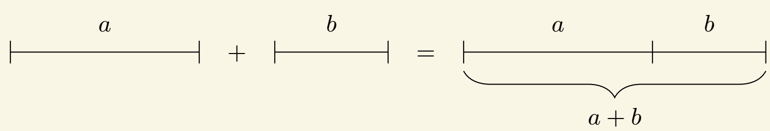
4 Which of the calculations did you find to be easier with this technique? Which were harder? What are the key differences that make one easier than the other?

25.4 Addition - Worksheet 4

1 We are going to spend this time looking at numbers from the Greek perspective. This means that we're going to think of numbers as being sticks of specific lengths. With this in mind, addition of numbers is represented by finding the total length of two sticks put end-to-end.

One of the reasons that the Greeks liked this framework is because it allows us to work with abstract ideas about arithmetic rather than actually having to measure out physical lengths. We can simply replace the numbers with variables and the picture remains meaningful.

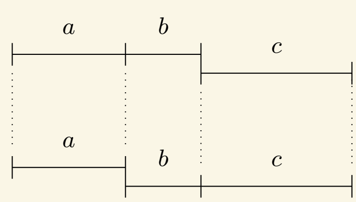
The units can be whatever you want them to be: inches, feet, centimeters, miles, . . .



Draw a diagram to represent the calculation $b + a$ and compare your diagram to the diagram of $a + b$. Explain why the two end results are the same length. Which property of addition is being demonstrated by this?

You may find it hard to put this into words. Try your best. It might help to think about what would go wrong if $a + b \neq b + a$.

2 Using this framework, the meaning and validity of the associative property of addition is made much more apparent as well.

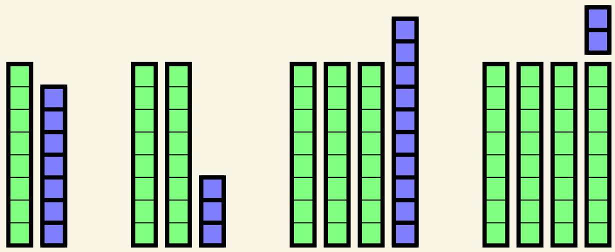


Determine which diagram represents $(a + b) + c$ and which one represents $a + (b + c)$. Explain how you reached your conclusion.

The purpose of this problem is to show you how the right diagram can help to communicate complex ideas in a simple manner.

25.5 Addition - Worksheet 5

1 Recall that the base-8 number system is a system where we work in groups of 8 instead of groups of 10. The concept of addition carries over perfectly as long as the group sizes are respected.



$17_8 \quad + \quad 23_8 \quad = \quad ??_8 \quad = \quad 42_8$

Draw a base-8 blocks diagram to represent $25_8 + 34_8$ and compute the result.

2 Draw a base-8 blocks diagram to represent $134_8 + 155_8$ and compute the result.

3 Think about (but do not draw) the diagram you would need to represent $34_8 + 17_8$ and compute the result from that mental picture.

25.6 Deliberate Practice: Addition Practice

Focus on these skills:

- Do the problem mentally before drawing any part of the number line diagram.
- Draw out the steps of adding the different place values on the number line diagram.
- Present your work legibly.

Instructions: Perform the given calculation mentally, then draw out a number line diagram to perform the calculation.

1. $43 + 25$
2. $18 + 38$
3. $26 + 49$
4. $76 + 19$
5. $54 + 21$
6. $35 + 45$
7. $29 + 52$
8. $276 + 112$
9. $135 + 349$
10. $289 + 428$

25.7 Closing Ideas

Addition is the very first arithmetic operation that children learn. The idea of putting collections of objects together into a single collection fits perfectly with the idea of counting. But as the collections of objects become larger, we are forced to find different ways to organize that information, and that's where many of the concepts we have for numbers come from. It's all about organizing information in a useful way.

Once we have the framework in place, we can then move on to see how the organization is helpful. As we discussed earlier, adding in columns used to be an important organizational framework when all of these calculations were done by hand. And it is that framework that leads us to ideas like "carrying the one." In today's world, that specific way of organizing information is less useful. This doesn't mean that it's wrong or that it can't still be used. But we are seeing that there is more value in helping students develop flexible ways of thinking rather than focusing on computational algorithms.

Being able to think about numbers in different bases is important if you are interested in learning more about how computers work, especially if you want to get into programming. Being able to work in the framework of the number line helps with any job that uses geometric ideas, including many forms of design. And those are the reasons why we're taking the time to explore these concepts here. Even though we've been discussing addition, we're really just building a framework to help you work with numbers in whatever context you will see them in the future.

All of the information in a computer is stored as binary digits. The color codes on webpages are stored as hexadecimal values.

As we continue to explore mathematical thinking, it will be helpful to take time to reflect on the new ways of looking at math that you're seeing. A lot of students struggle with their confidence when it comes to math, and that's usually the result of being told that there is only one way of doing things that doesn't really connect with them. That is an unfortunate legacy of the current math education system. But it's never too late to start something new. Hopefully, the approaches that you see in the coming sections will provide you with an opportunity to overturn past negative experiences and help you to see both the logic and the beauty of mathematics.

25.8 Going Deeper: Automaticity with Arithmetic

When it comes to arithmetic, faster is not always better. In fact, there a number of people who are not naturally fast at arithmetic, at yet have become very talented mathematicians. However, this does not mean that there is no value at all to being able to do so arithmetic moderately quickly and confidently.

There are also some mathematicians who are amazingly fast at arithmetic, but have only become that way through intensive practice. For example, look for a video of Arthur Benjamin doing "mathemagic" on the internet.

It is easiest to understand this concept by using an analogy with reading. Consider the word cat. As an adult, you probably see the word all at once and can recognize that it's the word cat (as opposed to dog or catastrophe). But let's think about what goes into learning the word cat as a child. For young children, a common reading technique is to "sound out" the various letters. Rather than seeing cat all at once, it's seen as "c - a - t" (with each letter sounded out individually). And after making the sounds in faster and faster succession, the child will eventually understand that it is the word cat.

There's an interesting thing that happens to the brain with reading as people get better at it. It actually starts taking less and less brain power to read as you do more of it. The child that is sounding out the individual letters of the word cat is investing a significantly larger part of their brain than an adult that immediately recognizes it. Basically, our brains get used to seeing the word so often that we have a mental shortcut that allows us to identify it right away.

Short words like cat are known as "sight words" because they are words that we eventually expect children to recognize instantly.

This skill doesn't make us speed readers, but it does allow us to read fluently. As sentences become longer and their meanings become more complex, the fact that our brains don't have to work hard to recognize words gives us more brain space to think about the ideas rather than using it all up just figuring out which words are on the page.

A similar thing happens with mathematics. Students that struggle with basic arithmetic and algebra are often unable to take in the larger mathematical ideas because their brains are bogged down in the calculations. But even a moderate level of mathematical fluency creates space for students to start to have mathematical ideas and make important connections.

The ability to immediately recognize basic arithmetic facts and fluidly perform simple algebraic manipulations is known as *automaticity*. Mathematical automaticity is the mathematical equivalent of sight words. It's when you can recognize that $7 + 4 = 11$ and $6 \cdot 9 = 54$ without needing to do a bunch of counting or other mental manipulations.

There is a thin line between the practice required to develop automaticity and what is often called "drill and kill" (the endless repetition of calculations that leads to the destruction of all motivation). To understand this, we can draw from another analogy, but this time with music. In the context of learning a musical instrument, there are a core set of exercises (scales and arpeggios) that students practice. These exercises are rarely an end for themselves. That is, nobody goes to concerts to listen to someone simply play scales. However, fluency with scales and arpeggios leads to greater skill in playing more complex music, as the complex music is built out of scales and the intervals found in arpeggios. Another way of saying this is that those exercises help to build the musician's musical vocabulary, so that they are better able to "understand" the music that they're learning.

The same is true for mathematics. The reason math teachers hope for students to reach a certain level of fluency with basic calculations is that our larger mathematical ideas are often built on those calculations. Basic arithmetic is required for almost every pursuit within mathematics, and almost every mathematical idea is built from some experience that can be grounded back in our experiences of basic arithmetic.

As hinted at above, the development of automaticity requires regular practice. Fortunately, the amount of practice is not very significant. Flash cards and online arithmetic practice programs are readily available, and there are cheap or free options. All it takes for many people is just a few minutes a day for a couple weeks to reach a reasonable level of proficiency.

Awareness of your common errors helps you to recognize those situations as they arise, which helps you to fix it.

What skills should you practice? Interestingly, it's the same basic arithmetic drills that we would use with elementary school children:

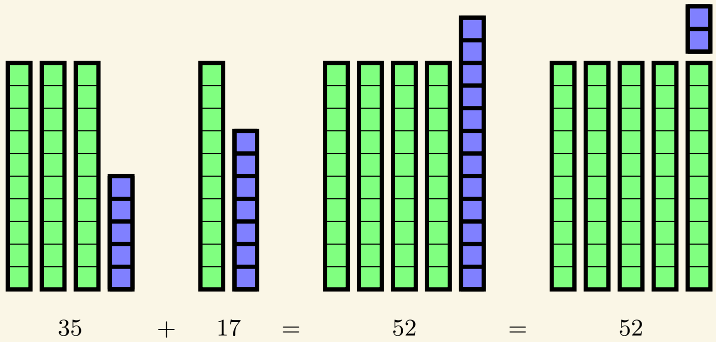
- Addition: One-digit plus one-digit
- Subtraction: One-digit or two-digit minus one digit
- Multiplication: One-digit multiplied by one-digit
- Division: The inverses of the multiplication problems (If you have the multiplication problem $5 \cdot 9 = 45$, then the inverse problems are $45 \div 9 = 5$ and $45 \div 5 = 9$.)

You are encouraged to keep a record of your progress. You might be quite surprised to see how much progress you make in a short period of time with just a little bit of practice. These exercise has been used with college students, and they have reported that their overall level of mathematical confidence has risen as a result of doing them. And in many senses, that is one of the most important obstacles that students who are struggling with mathematics can overcome. The pattern of mathematical self-doubt that a wide range of students bring with them to college is something that can hold them back from accomplishing their goals, and it's incredible how a simple exercise like this can reap positive benefits.

In the "Going Deeper" for the following section, we'll talk about how to build larger mental arithmetic skills on the foundation of this basic arithmetic automaticity.

25.9 Solutions to the “Try It” Examples

1



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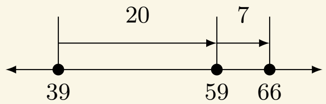


$39 + 27 = 66$

3

$271 + 119 = 390$

4 $183 + 319 = 502$

Borrowing is Overrated: Subtraction

Learning Objectives:

- ☐ Understand what it means to “borrow” in subtraction.
- ☐ Understand how subtraction can be visualized on a number line.
- ☐ Develop effective strategies for mental subtraction for 2-digit and 3-digit numbers.

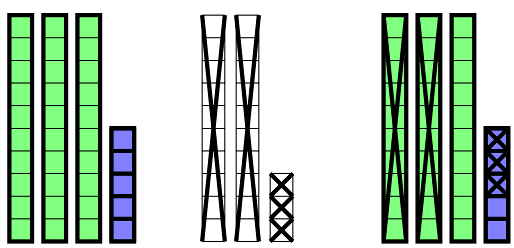
The first concept of subtraction that students learn is the idea of “taking away” objects from a collection. In fact, many children learn to speak the subtraction calculation $5 - 3$ as “5 take away 3.” Here is one way to represent this graphically:

$5 \quad - \quad 3 \quad = \quad 2$

Think of the boxes with an X over them no longer existing in the collection.

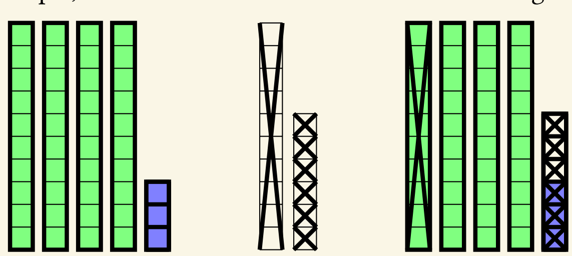
Subtraction is interesting because the “taking away” step isn’t an object, but an action. We have used an X to symbolize that idea because it’s visually easy to think about the final picture with parts crossed off. But that is not to say that the X is itself an object. It doesn’t “count” towards the final tally. It is an action that is performed on the collection.

The extension of this idea is to do it with base-10 blocks. This mixes the concept of “taking away” with our method of organizing numbers.



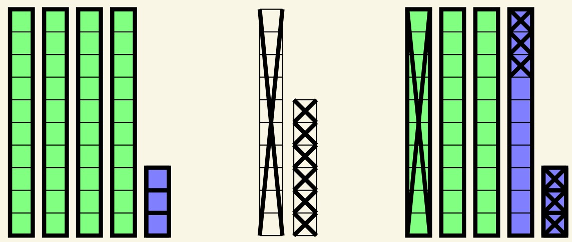
$35 \quad - \quad 23 \quad = \quad 12$

1 Here’s another example, but this time we will encounter something different.



$43 \quad - \quad 16 \quad = \quad ??$

We can see that we still need to remove some unit cubes, but we don’t have any unit cubes left. So we will convert one of the rods into 10 unit cubes so that we can finish removing the pieces.



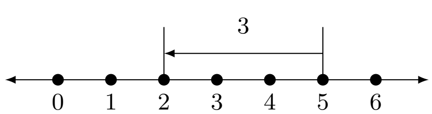
43 − 16 = 27

The step of converting a rod into individual unit cubes is an example of the "borrowing" step for subtraction. We reduced the number of rods by 1 and increased the number of unit cubes by 10.

Try it: Draw a base-10 blocks diagram to represent $42 - 28$ and compute the result.

Just like "carrying the one," the step of "borrowing" is a bookkeeping step. When subtracting in columns, if you don't cross out that number, you might forget that you borrowed and then make a mistake down the line. This is why the process was so heavily emphasized. But you don't actually have to do it that way at all. There are natural ways to keep track of that information. We're going to once again look at the number line for insight.

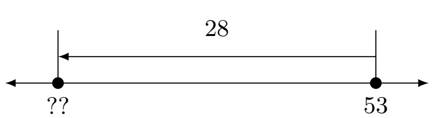
One way to look at subtraction on the number line is to think about movement. But while addition moved to the right, subtraction moves to the left. We will start with a diagram of $5 - 3$:



$\boxed{5 - 3 = 2}$

- Step 1: Start at 5.
- Step 2: Move 3 spaces to the left.
- Step 3: End up at 2.

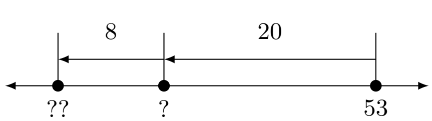
For small numbers, counting out the individual steps is reasonable. But for larger values, we're going to employ an organization technique similar to what we did with addition. Here is the setup for $53 - 28$: For small numbers, it is easy enough to just count out the steps. But for a larger calculation, counting is simply too slow. And in order to simplify the diagram, we're going to use just the part of the number line that's relevant. Here is the setup to calculate $37 + 46$:



$\boxed{53 - 28 = ??}$

- Step 1: Start at 53.
- Step 2: Move 28 spaces to the left.
- Step 3: End up at ??.

Just as before, we're going to break this out into two steps.



$\boxed{53 - 28 = ??}$

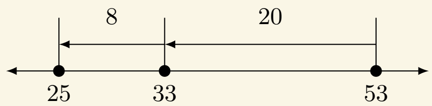
- Step 1: Start at 53.
- Step 2: Move 20 spaces to the left.
- Step 3: Move 8 spaces to the left.
- Step 4: End up at ?.

Once again, by simply applying a bit of mental thought, you can determine the unknown values. The value of ? is 33 and the value of ?? is 25.

Just as "carrying the one" represented an intuition of increasing numbers, "borrowing" is a representation of an intuition of decreasing numbers. Starting from the number 33 and moving to the left, you will basically never make the mistake of somehow increasing the value to 35. Your brain can do this on its own because you have enough experience with numbers.

What is $1000 - 1$? Did you have to do a lot of borrowing to get the answer, or did your brain just "know" that the number before 1000 is 999? This is that numerical intuition kicking in again.

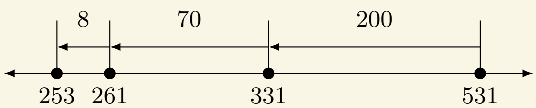
2 Here is the full diagram for the calculation $53 - 28$ using the number line.



$\boxed{53 - 28 = 25}$

Try it: Calculate $62 - 17$ using a number line.

3 Larger numbers can be done in the same way.



$\boxed{531 - 278 = 253}$

Try it: Calculate $623 - 376$ using a number line.

Most students find subtraction a little bit harder than addition. This is perfectly normal. Practice helps to close this gap.

4 Just as with addition, the framework of subtraction on the number line makes it less difficult to do mental subtraction compared to subtraction in columns. This is because you are relying on your intuition of numbers instead of having to actively remember whether you borrowed.

Try it: Calculate $572 - 158$ mentally.

Be methodical and organized.

26.1 Subtraction - Worksheet 1

1 Draw a base-10 blocks diagram to represent $31 - 18$ and compute the result.

2 Draw a base-10 blocks diagram to represent $317 - 158$ and compute the result.

3 Think about (do not draw) the diagram you would need to represent $36 - 19$ and compute the result from that mental picture. Did you find the visualization helpful or distracting? Explain what was helpful or distracting about the mental image for you.

The only correct answer to the “helpful or distracting” part of the question is an answer that is true to your own experience.

26.2 Subtraction - Worksheet 2

1 Calculate $57 - 35$ using a number line.

2 Calculate $324 - 158$ using a number line.

3 Think about (do not draw) the number line diagram you would need to calculate $42 - 26$ and compute the result from that mental picture. Do you prefer this visualization or the base-10 blocks visualization? Why?

4 Practice your mental arithmetic by performing the following calculations.

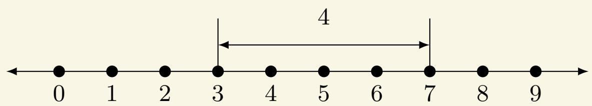
$41 - 27 =$	$53 - 38 =$	$39 - 25 =$
$82 - 54 =$	$67 - 48 =$	$43 - 37 =$
$344 - 257 =$	$401 - 235 =$	$518 - 276 =$

You might be surprised how much a little bit of practice can help to develop your skill and your confidence.

26.3 Subtraction - Worksheet 3

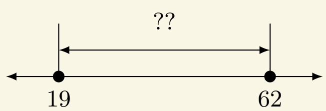
1 Instead of thinking about motion, subtraction can also be thought of as a distance between numbers. Here is a diagram that shows that $7 - 3 = 4$.

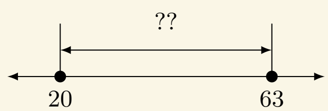
This works when the first number is to the right of the second number, otherwise you have to use negative distances.



$\boxed{7 - 3 = 4}$

One of the values of this idea is that the distances do not change if you shift both numbers the same amount. This can sometimes allow you to think of the calculation in a slightly different manner that makes it easier to calculate. Here are two calculations where one is just slightly shifted from the other.





What were the two calculations? Which of the two calculations was easier to do? Why?

2 There are two reasonable ways to calculate $201 - 149$ by shifting the values. Draw the corresponding diagrams and compute the result using both approaches. Which one was easier for you? Why?

3 Practice your mental arithmetic by performing the following calculations.

You are encouraged to explore using a mixture of the techniques you've seen. Some methods are better suited to some calculations compared to others.

$49 - 28 =$	$62 - 49 =$	$55 - 37 =$
$199 - 153 =$	$303 - 178 =$	$432 - 285 =$

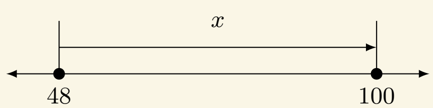
26.4 Subtraction - Worksheet 4

1 There is a way to leverage your addition experience for doing subtraction problems. It requires reframing the idea of subtraction as solving an addition algebra problem. Notice that the following two equations are equivalent to each other.

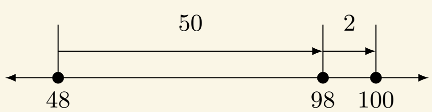
This is how cashiers used to count back change to people when cash was a far more common payment method.

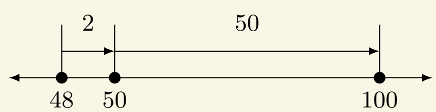
$$x = 100 - 48 \qquad\qquad 48 + x = 100$$

The first question asks, “What is the result of subtracting 48 from 100?” The second question asks, “48 plus what number is equal to 100?” While the answers will be the same, they represent two different approaches. We will focus on the second one. Here is a diagram for that question:



Rather than trying to count down from b by the amount a, this is now about counting up from a to b. The application of this is most common when the b value is a “nice” value to work from. The reason is that it’s mentally easier to break it down into different parts using the place values as a guide. Here are two diagrams for $100 - 48$, each showing a different visualization:





Which of the two calculations at the bottom is more intuitive for you?

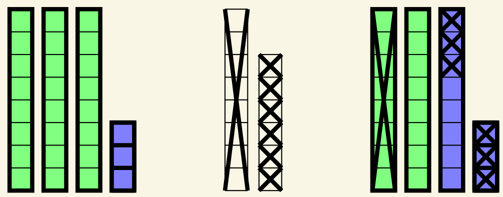
2 Mentally apply the above method of subtraction to perform the following calculations.

This is sometimes called the “counting up” method.

$100 - 28 =$	$100 - 77 =$	$100 - 31 =$
$150 - 48 =$	$150 - 87 =$	$150 - 95 =$
$300 - 153 =$	$800 - 418 =$	$600 - 238 =$
$1000 - 275 =$	$1000 - 318 =$	$1000 - 444 =$

26.5 Subtraction - Worksheet 5

1 In the same way that we used base-8 blocks to visualize addition in base-8, we can use it to help us perform subtraction.



$33_8 \quad - \quad 16_8 \quad = \quad 15_8$

Draw a base-8 blocks diagram to represent $42_8 - 25_8$ and compute the result.

2 Draw a base-8 blocks diagram to represent $143_8 - 55_8$ and compute the result.

3 Think about (but do not draw) the diagram you would need to represent $52_8 - 33_8$ and compute the result from that mental picture.

26.6 Deliberate Practice: Subtraction Practice

Focus on these skills:

- Do the problem mentally before drawing any part of the number line diagram.
- Draw out the steps of subtracting the different place values on the number line diagram.
- Present your work legibly.

Instructions: Perform the given calculation mentally, then draw out a number line diagram to perform the calculation.

1. $48 - 23$
2. $75 - 38$
3. $62 - 15$
4. $53 - 47$
5. $61 - 28$
6. $83 - 71$
7. $48 - 29$
8. $378 - 227$
9. $283 - 149$
10. $314 - 178$

26.7 Closing Ideas

In some ways, subtraction is very simple. You start with some objects and you take some away. But we have already seen that there are a number of ways of looking at how we might do this. We have used base-10 blocks (and base-8 blocks, if you did Worksheet 5) and we used the number line. But we also used the number line in several different ways. We saw subtraction as movement, subtraction as a distance, and subtraction reframed as addition.

But there's even more to subtraction that this. What happens if you are being asked to "take away" more than what you started with?

$$3 \quad - \quad 5 \quad = \quad ??$$

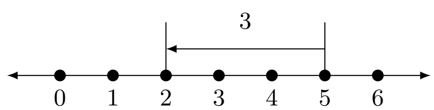
If you try to think about how an early elementary school student might respond, the most common explanation of this is "You can't do that!" As adults, we have other ways of thinking about this that can help us resolve the question, but it also requires a certain level of intellectual sophistication. Whatever explanation we might have is going to be more complex than just saying that you start with a collection and "take away" things from it.

All of this only goes to show that sometimes simple and easy are not synonymous. The idea of subtraction is fairly simple. But as you look deeper at it, the simplicity of it requires some rather complex ideas to fully understand. And this helps us to see the beauty of mathematics. Simple concepts can lead to complex ideas if you simply ask the right questions.

26.8 Going Deeper: Subtraction as Displacement

In Worksheet 3 of this section, we looked at the idea of using subtraction to measure distance. We're going to push that idea further and look at the signed distance between numbers, and see some applications of the idea.

Let's look at the calculation $5 - 3 = 2$. Using motion on the number line, we can interpret this as starting at 5 and moving 3 to the left to get to 2.

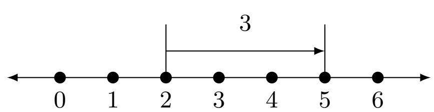


$$\boxed{5 - 3 = 2}$$

- Step 1: Start at 5.
- Step 2: Move 3 spaces to the left.
- Step 3: End up at 2.

However, we can get another interpretation of this by inverting the movement and turning it into a question: If we're starting at 2 and ending up at 5, what is the movement that we must make? And by looking at the picture, we can see that we need to move 3 to the right.

How do you get from 2 to 5? By moving 3 to the right.



$$\boxed{5 - 2 = 3}$$

Displacement is the movement from one position to another, and it can be expressed in terms of subtraction. To move from a to b, the required displacement is $b - a$. We can check this algebraically:

$$\underbrace{a}_{\text{start}} + \underbrace{(b-a)}_{\text{displacement}} = \underbrace{b}_{\text{end}}$$

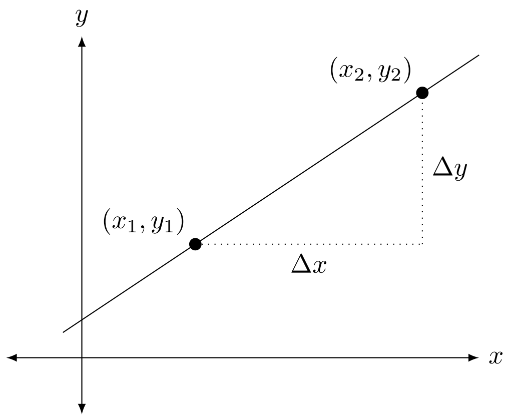
This concept gets applied in a number of ways at many different levels of mathematics.

In Section 13, we looked at the concept of slope. The main definition we presented was purely geometric:

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}} = \frac{\text{The change of } y}{\text{The change of } x}$$

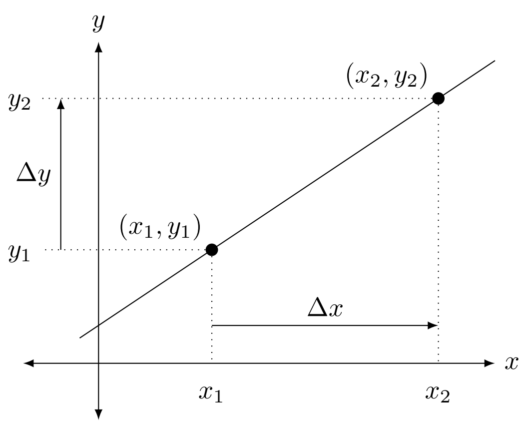
We're going to take a closer look at this idea through the lens of displacement to understand the algebraic formula for slope.

Consider a line that is passing through the indicated points (x_1, y_1) and (x_2, y_2) as shown below:



Notice that both Δx and Δy can be interpreted as displacements. They describe a motion to get from one location to another. Furthermore, these two displacements are independent of each other, meaning that as we change our x-coordinate, the y-coordinate is fixed, and the same is true the other way around. In fact, we can look at both of these displacements relative to the corresponding axes:

The technical term for viewing each of these displacements relative to their axes is a *projection.*



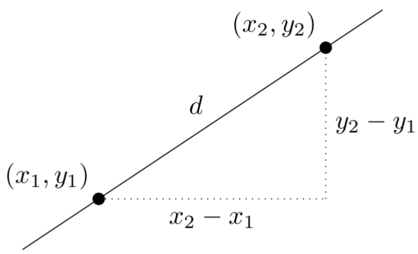
By viewing the movements in this way, we can see that we are looking at displacement on the number line, and can use the formula for displacement for each to get the algebraic formula for slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This formula was given in a margin comment in Section 13, but we didn't explain it at the time because the focus was on the geometric interpretation and building basic intuition and experience. But now that we have the concept of displacement, this formula is not as mysterious.

As you continue forward in mathematics (and also science and statistics classes), you will see lots of situations where subtraction is being used to represent a quantity like acts like a displacement (though the language will vary depending on the topic). More generally, these concepts appear in vectors, which can be used to represent displacement in multiple dimensions. The following are a few examples of this. Don't worry if you don't understand these, as they're not relevant to this class. But if you've seen them before, it will help you to put this idea into context.

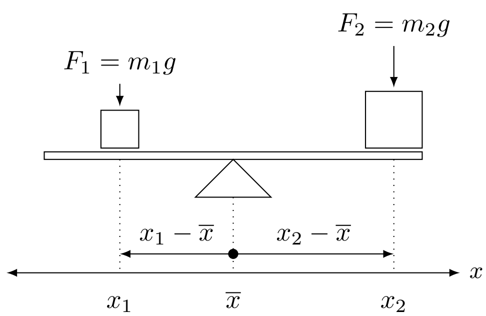
- The Distance Formula: The distance formula is the Pythagorean Theorem applied to points on a plane. The lengths of the two sides of the right triangle are found using the same Δx and Δy concepts that we used with slope.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Torque: Torque is the result of applying a force around a pivot point. The torque is equal to the force multiplied by the displacement from the pivot. The location of the objects is often given relative to an underlying coordinate system. In fact, it is this idea that leads to the formula for the center of mass of an object.

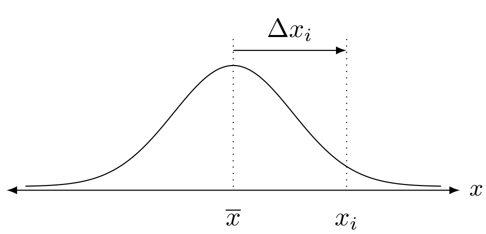
This description is valid when the force is perpendicular to the lever arm. Otherwise, it's a little more complicated.



$$\tau = F_1(x_1 - \overline{x}) + F_2(x_2 - \overline{x})$$

- Deviation from the Mean: In statistics, we often want to know where a particular data point is located relative to the mean of the sample or population. Concepts like this relate the standard deviation, z-scores, and correlations.

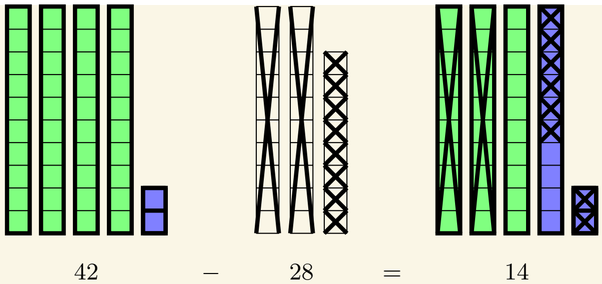
It is no coincidence that the notation $\overline{x}$ was used in both this example and the previous one. They can both be used to represent an average if set up properly.



The more familiar you are with the basic structures of mathematics, the more capacity you will have for understanding mathematical applications. Instead of seeing collections of random symbols, you will start to see concepts and ideas, which will help everything make more sense.

26.9 Solutions to the “Try It” Examples

1



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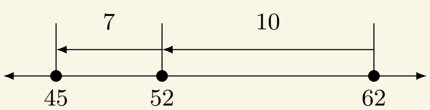
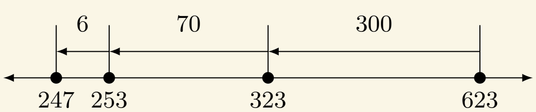


$62 - 17 = 45$

3



$623 - 376 = 247$

4

$572 - 158 = 414$

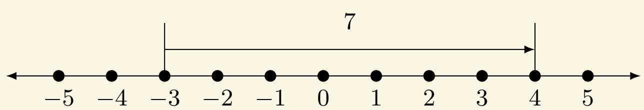
Through the Looking Glass: Negatives on the Number Line

Learning Objectives:

- ☐ Understand how to visualize adding to a negative number on a number line.
- ☐ Understand how to visualize subtracting from a negative number on a number line.

When we introduced the number line, we showed that the negative numbers were the numbers to the left of zero. But we have not yet worked with negative numbers in our arithmetic.

1 It turns out that if we add to a positive number or subtract from a positive number, there are no conceptual changes to how we work with the number line. Here is an example for addition. Notice that the process has not changed.

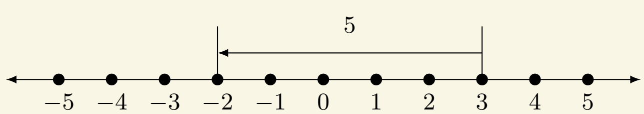


$\boxed{-3 + 7 = 4}$

- Step 1: Start at -3.
- Step 2: Move 7 spaces to the right.
- Step 3: End up at 4.

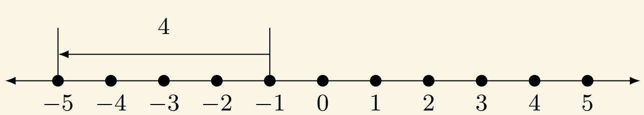
Try it: Calculate $-8 + 4$ using a number line.

2 Here are two subtraction calculations. Notice that it does not matter whether we start from a negative value or simply cross over from positive values to negative values. In both situations, the process remains exactly the same.



$\boxed{3 - 5 = -2}$

- Step 1: Start at 3.
- Step 2: Move 5 spaces to the right.
- Step 3: End up at -2.



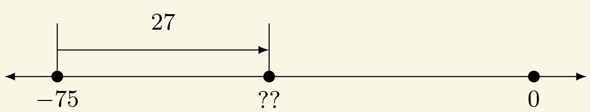
$\boxed{-1 - 4 = -5}$

- Step 1: Start at -1.
- Step 2: Move 4 spaces to the right.
- Step 3: End up at -5.

Try it: Calculate $2 - 7$ using a number line.

At this point, it is important that the number we are adding or subtracting is positive, and that all we are doing is allowing negatives as the starting position. We will look at the other possibilities in another section. But as long as that restriction is met, we can expand our ideas to allow us to think about addition and subtraction with larger numbers.

3 Let's consider the calculation $-75 + 27$. When we try to draw this on a number line, there are two observations we need to make. The first is that we are starting at a negative value, so we must begin our movement to the left of zero. The second observation is that the movement to the right is smaller than the distance to zero. This means that we will stay to the left of zero when we are done.

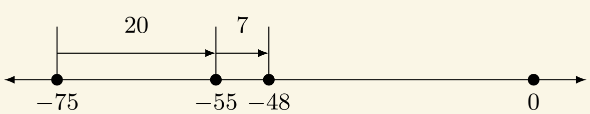


$\boxed{-75 + 27 = ??}$

- Step 1: Start at -75.
- Step 2: Move 27 spaces to the right.
- Step 3: End up at ??.

From here, it doesn't matter how you get to your answer. If you need to do the calculation in two steps, then do it in two steps. If you can perform the calculation mentally, that's also fine. But avoid using a calculator. It is worth the time to practice your mental arithmetic skills, as this feeds directly into your general mathematical confidence. The important thing is to think about the relative locations of the numbers.

Do not add or subtract in columns with negative numbers. The algorithms you've learned will likely lead to errors.



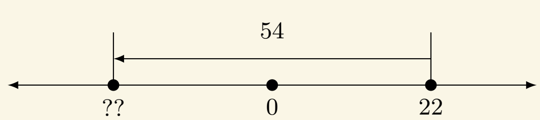
$\boxed{-75 + 27 = -48}$

- Step 1: Start at -75.
- Step 2: Move 20 spaces to the right.
- Step 2: Move 7 spaces to the right.
- Step 3: End up at -48.

This is mostly about the practice of thinking through the diagram to understand the underlying logic, not about following rules. It would be possible to write out a set of 5 or 6 rules for how to do all of these problems, but if you simply think about the picture, many of those rules will work out naturally and intuitively.

Try it: Calculate $-53 + 38$ using a number line.

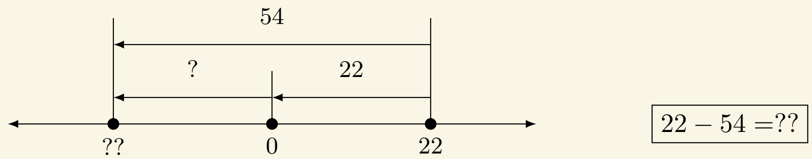
4 Sometimes, the calculation takes us past zero. In this case, we just need to look at the picture and think about what that means. Consider $22 - 54$. If we start at 22 and move 54 spaces to the left, we have definitely gone past zero.



$\boxed{22 - 54 = ??}$

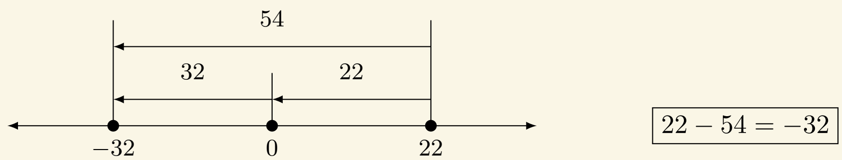
- Step 1: Start at 22.
- Step 2: Move 54 spaces to the left.
- Step 3: End up at ??.

Conceptually, it's easiest to think of this as simply returning to zero and then taking the remaining number of steps to finish the movement.



- Step 1: Start at 22.
- Step 2: Move 22 spaces to the left.
- Step 2: Move ? spaces to the left.
- Step 3: End up at ??.

If you need to move a total of 54 steps to the left, and you've already moved 22 steps, how many steps are left? You should hopefully be able to think that through and figure out that there are 32 steps remaining. And that leads us to the final result.



Try it: Calculate $34 - 78$ **using a number line.**

5 You do not want to think about these calculations as rules. You just want to think about them as following the natural logic of the problem. Draw the diagram and let the diagram guide your thinking.

Try it: Calculate $-15 - 32$ **using a number line.**

27.1 Negatives on the Number Line - Worksheet 1

1 Calculate $-3 + 5$ using a number line. Draw all the points and count out the steps.

2 Calculate $3 - 8$ using a number line. Draw all the points and count out the steps.

3 Calculate $-3 - 4$ using a number line. Draw all the points and count out the steps.

4 Without performing the calculation, explain why $387 - 749$ will result in a negative number.

5 Without performing the calculation, explain why $-178 - 455$ will result in a negative number.

27.2 Negatives on the Number Line - Worksheet 2

1 Calculate $19 - 45$ using a number line.

2 Calculate $-28 - 46$ using a number line.

3 Calculate $-15 + 73$ using a number line.

4 Practice your mental arithmetic by performing the following calculations without drawing a number line (though you may certainly visualize one).

$33 - 57 =$	$-38 + 19 =$	$-32 - 29 =$
$-42 + 70 =$	$-43 + 37 =$	$72 - 34 =$
$38 - 14 =$	$-52 + 31 =$	$-22 + 49 =$
$-18 - 35 =$	$28 - 63 =$	$-24 + 15 =$

27.3 Negatives on the Number Line - Worksheet 3

1 Calculate $159 - 217$ using a number line.

2 Calculate $-228 - 146$ using a number line.

3 Calculate $-215 + 273$ using a number line.

4 Practice your mental arithmetic by performing the following calculations without drawing a number line (though you may certainly visualize one).

These are difficult mental calculations! Aim for being accurate, not fast.

$-211 - 157 =$	$-118 + 205 =$	$-312 + 129 =$
$-138 - 124 =$	$-152 + 361 =$	$-212 + 188 =$
$-242 + 170 =$	$-143 + 137 =$	$238 - 314 =$

27.4 Negatives on the Number Line - Worksheet 4

1 Calculate $-53 + 27$ using a number line.

2 Earlier in the section, there was a warning about adding and subtracting in columns when working with negative numbers. We are going to explore the challenges that arise in this setting in order to more fully understand the challenges that arise from working in columns.

Below a possible first step (ones column) of the calculation $-53 + 27$ when performed using columns:

$$\begin{array}{r} {}^{1} \\ -53 \\ +27 \\ \hline 0 \end{array} \qquad \begin{array}{c} 3 + 7 = 10 \\ \text{Carry the one} \end{array}$$

Identify the error that has already taken place in this calculation.

This can be difficult. If you do not get an answer in a few minutes, just go on to the next problem.

3 There is a "rule" that can be used for doing this calculation in columns. To calculate $-a + b$ (where a and b are positive numbers).

- Step 1: Identify the larger number.
- Step 2: Perform the calculation "larger minus smaller."
- Step 3: Give your result the same sign as the larger number in the original problem.

Apply this "rule" to the calculation $-53 + 27$. Explain how this "rule" is a more complicated expression of the number line calculation.

27.5 Negatives on the Number Line - Worksheet 5

1 In the previous worksheet, we saw that calculations in columns can be problematic and lead to errors. The "rule" that was provided is the common way that this is taught. But this is not the only way to think about doing this calculation in columns. We're going to explore this in a different way.

Rather than working with digits, we will work with values. We will look at the calculation $-53+27$ again, but rewriting the calculations using expanded form. From here, it is much easier to perform the calculation while avoiding errors.

$$\begin{array}{r} -53 \\ +27 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{rr} -50 & -3 \\ +20 & +7 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{rr} -50 & -3 \\ +20 & +7 \\ \hline -30 & +4 \end{array} = -26$$

Notice that this presentation does not conform to any of the best practices of presentation. This should really be considered as scratch work.

Using the expanded form version of writing the calculation, calculation $42 - 76$.

2 What this is showing is that the calculation can be performed if we focus on individual place values. Upon a deeper investigation, this would also reveal that the real issue comes down to the steps of "carrying the one" or "borrowing." The digit manipulations that one might normally do are incompatible with the algorithms for addition or subtraction in columns. Here are two of the most reasonable attempts at performing this calculation using the traditional algorithms.

$$\begin{array}{r} \overset{1}{-53} \\ +27 \\ \hline -20 \end{array} \qquad \begin{array}{c} 3+7=10 \\ \text{Carry the one} \\ \\ 1-5+2=-2 \end{array} \qquad\qquad \begin{array}{r} -53 \\ +27 \\ \hline -34 \end{array} \qquad \begin{array}{c} -3+7=4 \\ \\ -5+2=-3 \end{array}$$

As best as you can, try to explain the conceptual errors of each attempt.

This is a very tricky problem! It's not obvious to a lot of people what is going wrong here. Try using the previous problem to help your thinking.

27.6 Deliberate Practice: Practice with Negatives

Focus on these skills:

- Do the problem mentally before drawing any part of the number line diagram.
- You are free to draw as few or as many steps in your number line calculation as you find necessary.
- Present your work legibly.

Instructions: Perform the given calculation mentally, then draw out a number line diagram to perform the calculation.

1. $-34 + 18$
2. $23 - 49$
3. $55 - 27$
4. $-48 + 32$
5. $-32 + 58$
6. $35 - 71$
7. $45 - 18$
8. $-328 + 455$
9. $243 - 345$
10. $-314 + 146$

27.7 Closing Ideas

In this section, we took a concept that is traditionally very confusing for students and presented it as a single image that most people find simple and intuitive. We are once again leveraging both your intuition and experience with numbers to create a framework that is easy for you to understand and use. But this really isn't about the calculations. It's about the idea of the calculation. We are in the process of exploring just how far we an go with addition and subtraction using the number line. What we have shown in this section is that the number line is a more powerful idea than the addition and subtraction algorithms.

And that is the general situation in mathematics. Very often, a good idea will take you further than just applying formulas or prescribed algorithms. Developing a depth of mathematical thinking can help you solve problems that you've never seen before by relating it back to things that you have. These calculations with negative numbers are extremely accessible and understandable without needing to do a lot of extra work because we were able to relate it back to the number line, which we have had plenty of experience with over the last few sections. This is far better than simply having you memorize a complex set of rules for adding and subtracting in columns.

Whenever possible, you should try to focus on ideas rather than rules. You will be able to go so much further if you do.

27.8 Going Deeper: The Absolute Value Function

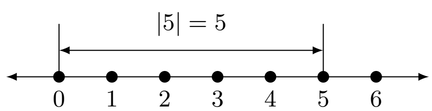
Consider the following question: Is $-1,000,000$ a bigger number than 1? The answer you give depends on how you interpret the notion of size. For example, we can say that $1 > -1,000,000$ by just thinking about the number line, and if we think that "bigger" is the same as "greater than" we can conclude that $-1,000,000$ is not bigger than 1. On the other hand, if you think about "bigger" as meaning the "size" of a number, then we might conclude that $-1,000,000$ is a pretty big number relative to 1. The notion of the "size" of a number is not captured by the inequalities we used earlier, and so we need to introduce a new mathematical object.

The *absolute value* of a number has several different representations, and the choice of representation depends on the details of the situation. We will start with the geometric interpretation.

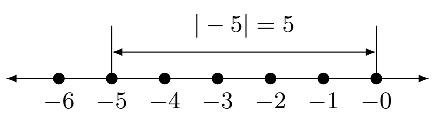
Definition 27.1. The *absolute value* of the number x is denoted by $|x|$ and is the distance between 0 and x on the number line.

Implicitly, the distance is a nonnegative value. If we need something like a negative distance, then it's better to use displacement.

Because of how the number line is constructed, it's easy to see that when $x > 0$, we have $|x| = x$. Here is an example of that:

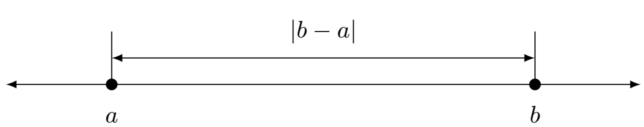


When we have a negative number, $|x|$ takes the value of x when we ignore the negative sign.



This framework for the absolute value should remind you of some of the ideas we used when we were working with subtraction. In fact, we can use this to define the distance between two points. The *distance* between the numbers a and b is $|b - a|$. Notice that this has the same value as $|a - b|$. You will see both definitions used, but the first definition draws a better connection between distance and displacement. In fact, sometimes distance is simply defined to be the absolute value of the displacement, and you're just expected to know how to compute the displacement.

If you pay close attention, you'll see that the definition of absolute value invokes the concept of distance, but then we later say that distance is the absolute value of displacement. This feels like circular reasoning, but it's actually a reflection of what concepts you're using as your assumptions. For our definition of absolute value, we're using your intuitive notion of distance as the starting point.



The algebraic definition is slightly more complicated because it requires a special type of notation:

Definition 27.2. The *absolute value* of the number x is denoted by $|x|$ and is given by the following formula:

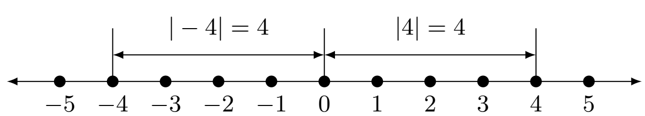
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

We first need to describe this notation. This is the notation that is used for a *piecewise defined function*, which is a function that uses different formulas depending on the value that it has been given. You can see that this definition is broken into two parts. The first formula is applied when $x \geq 0$ and the second formula is applied when $x < 0$. This means that you have to decide which category your number is in before you apply the formulas. If you take a precalculus course, you will probably run into this idea there and get a lot more experience. For now, it is enough to understand that this is a special way of building a function using multiple different pieces.

From this definition, we can see that this is the same as we had with the geometric definition. When the value of x is positive, then we have $|x| = x$. This is also true when $x = 0$. When x is negative, the geometric definition led us to ignoring the negative sign. Unfortunately, that concept doesn't have a good algebraic equivalent. However, it's computationally true that the negative of a negative number is a positive number, and so that's how we communicate it.

Working with equations involving absolute value can feel very different depending on whether you're thinking about it from the geometric perspective or the algebraic perspective. For example, let's consider the equation $|x| = 4$.

Geometrically, this is asking us to find the numbers that are a distance 4 from the 0. And we can immediately determine this by looking at the number line and moving 4 spaces in either direction starting at the origin.



This tells us that the solutions are $x = -4$ and $x = 4$. We can express this as $x = \pm 4$, where the $\pm$ symbol is a shorthand for two different possibilities.

In order to solve this from the algebraic perspective, we need to think about the two different situations. When $x \geq 0$, we have $|x| = x$, so that the equation becomes $x = 4$. On the other hand, if $x < 0$, then $|x| = -x$, so that our equation becomes $-x = 4$, which we can solve to get $x = -4$. We still arrived at $x = \pm 4$ as the solution, but it required us to work with two different cases. Here is one way of presenting that work:

$$|x| = 4$$

If $x \geq 0$:	If $x < 0$:
$x = 4$	$-x = 4$
	$x = -4$

The same ideas can be employed for more complicated equations. When you have mathematical expressions inside of the absolute value sign, you need to treat that as a single object.

$$|x+7| = 3$$

$$\begin{array}{ll} \text{If } x+7 \geq 0: & \text{If } x+7 < 0: \\ x+7=3 & -(x+7)=3 \\ x=-4 & x+7=-3 \\ & x=-10 \end{array}$$

Most textbooks do this calculation in a much more condensed presentation:

$$\begin{aligned} |x+7| &= 3 \\ x+7 &= \pm 3 \\ x &= -7 \pm 3 \\ x &= -4 \text{ or } -10 \end{aligned}$$

This is ultimately the same process, but it starts to mask some of the ideas and turns it more into an exercise of executing rote calculations rather than providing insight into how the absolute value function works. Since our focus is on the ideas, we've left it in the slightly longer form. Ultimately, you can do this however you choose.

There's a catch to the algebraic approach, which is that it is possible to go through the algebra and get an answer, but find that the answers don't satisfy the original equation. For example, let's look at $|x| = -4$. If we simply pushed ahead with the algebra, it would look like this:

$$|x| = -4$$

$$\begin{array}{ll} \text{If } x \geq 0: & \text{If } x < 0: \\ x=-4 & -x=-4 \\ & x=4 \end{array}$$

If we only looked at the last lines, we would say that $x = \pm 4$, just as before. But if we tried to plug those values in, we would find that they aren't actually solutions.

The problem is that the conclusion in each column is in contradiction with the assumption. On the left, we were supposing that x is a nonnegative number, but we concluded that $x = -4$. On the right, we have a similar problem. These are sometimes called *apparent* solutions (or sometimes *phantom* solutions) because they seem like they should be solutions even though they aren't.

There are a number of methods that you can employ to identify these situations. The first is to simply check that your solutions really work in the original equation. This is the most common approach that math textbooks teach. The reason this is often suggested is that it's straight-forward to perform all the calculations, and you don't really have to think very much about what's going on. You can simply grind out the calculations from beginning to end without really thinking about things. And from a practical perspective, there's nothing wrong with this approach. It will get the job done.

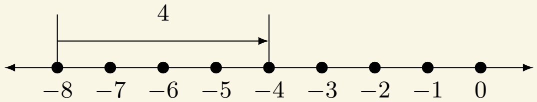
However, It can be helpful to practice being more thoughtful. By thinking ahead, you can sometimes save yourself the extra work by simply asking whether your absolute value equation is equal to a positive or a negative value. For the equation $|x| = -4$, we can immediately conclude that there are no solutions because the absolute value function will never give a negative value.

Ultimately, neither approach is inherently better or worse. For simpler equations, thinking ahead can save some work. But there are more complicated examples (such as when the variable is both inside and outside the absolute value) where it can be more work to think through the possibilities than to simply grind out the algebra and check to see if they work. Regardless of the approach, this is a feature that you will need to keep in mind when solving equations involving the absolute value signs.

(There is another set of complications that arise when working with absolute value inequalities. You end up needing to work with compound intervals and paying close attention to the exact form of your equation. These skills are sometimes taught in courses, but sometimes not. It usually ends up being another set of memorized algebraic rules, and it has limited value to students at this level. So we're just not going to go there.)

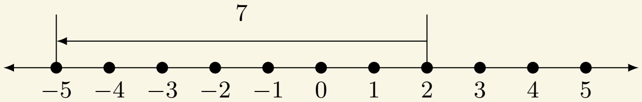
27.9 Solutions to the “Try It” Examples

1



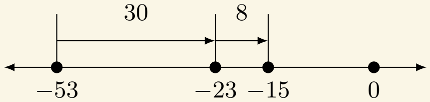
$$\boxed{-8+4=-4}$$

2



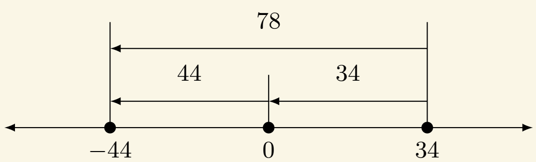
$$\boxed{2-7=-5}$$

3



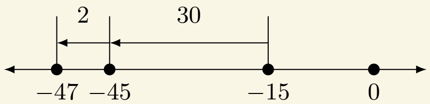
$$\boxed{-53+38=-15}$$

4



$$\boxed{34-78=-44}$$

5



$$\boxed{-15-32=-47}$$

When All the Chips are Down: Calculations with Integer Chips

Learning Objectives:

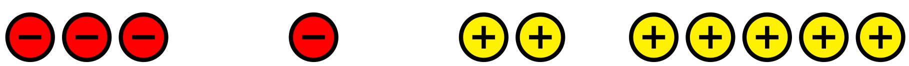
- ☐ Represent numbers using integer chips.
- ☐ Understand how to visualize addition using integer chips and zero pairs.
- ☐ Understand how to visualize subtraction using integer chips and zero pairs.

At the end of an earlier section, we talked about how certain subtraction problems do not work well with the basic idea of counting objects because you can be asked to subtract off a number that's larger than the number of objects that you have.

$3 \quad - \quad 5 \quad = \quad ??$

It turns out that we can still represent this idea, but we have to use a different type of manipulative. These are commonly called *integer chips*. These are plastic circles where the two sides are different colors (usually yellow and red). We are going to enhance these by also using a plus and minus sign in our diagrams. These objects allow for multiple alternative interpretations, such as electric charges or savings/debt. Here are some collections of chips and the numbers that they represent.

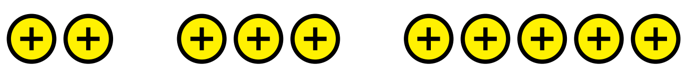
We can technically use the same symbols of a box and an X, but these objects are not as intuitive as the positive and negative chips.



$-3 \qquad -1 \qquad 2 \qquad 5$

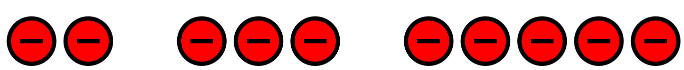
These are organized left-to-right in the same order as they would appear on the number line.

Notice that if we only focus on the positive chips, they behave exactly like we might expect.



$2 \quad + \quad 3 \quad = \quad 5$

If we were to add together negative chips, it turns out that the arithmetic would work out correctly as well.



$-2 \quad + \quad (-3) \quad = \quad -5$

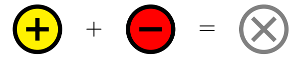
We put parentheses around the -3 because it is considered poor mathematical notation to have two arithmetic symbols immediately next to each other.

- Good: $-2 + (-3) = -5$
- Bad: $-2 + -3 = -5$

But what happens when we have a mixture of the two types of chips? Since one step left followed by one step right would leave you right back where you started, we can see that a positive chip and a negative chip will cancel each other out. And by eliminating those pairs, we can see what the final answer would be.

$-2 \quad + \quad 4 \quad = \quad 2$

We will show the canceled pairs explicitly.

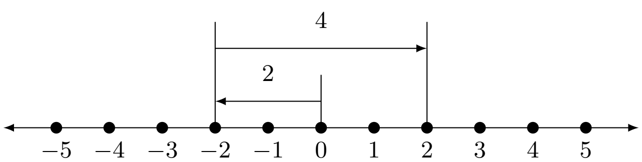


1

Try it: Calculate $4 + (-5)$ using an integer chip diagram.

As always, we want to avoid treating this as a system of "rules." We want to be able to understand these manipulatives as objects that represent familiar ideas. Specifically, we can relate these chips to movement on the number line. Thinking about the positive chips as movement to the right and negative chips as movement to the left, we can translate the calculation into a number line picture. This is a subtle shift from what we were looking at before. Initially, we were just starting from a position and making one movement. Now we're starting from zero and making two movements. But the primary concepts still remain the same.

-2	$+$	4	$=$	2
2 steps left		4 steps right		2 steps right



$\boxed{-2 + 4 = 2}$

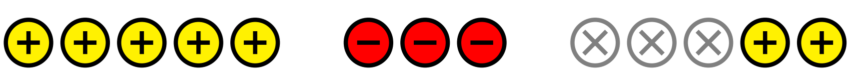
2

Notice that this framework allows us to add any two numbers together, which is slightly more than what we did on the number line in the previous section. For that, addition always meant movement to the right. But this picture shows us that adding a negative number means moving to the left.

Try it: Calculate $3 + (-5)$ using an integer chip diagram. Then draw a number line picture to represent the calculation as movement on the number line.

Now that we know how to represent addition of any two numbers with these chips, we will turn our attention to subtraction. Let's look at $5 - 3$. Instead of thinking of subtraction as "taking away" chips, we can get an equivalent form by flipping over the subtracted chips and using zero pairs.

$5 \quad - \quad 3$

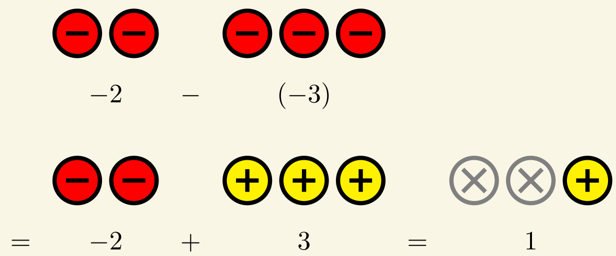


$= \quad 5 \quad + \quad (-3) \quad = \quad 2$

Without physical manipulatives, it can be easy to forget the step of flipping over the chips. It can be helpful to physically turn your hand over as if you're flipping the chips when thinking through these problems. Learning that happens with both mind and body is more easily remembered.

It's easy for us to verify that this gives us the same final answer as any other method we use. But what is happening here? What we have done is replace the concept of "taking away" with a form of "undoing" the steps. How do we "undo" three steps to the right? By taking three steps to the left! The chips are merely a way of representing that concept.

3 There were calculations that we were unable to do with our previous concepts, but we can now do with the integer chips. Here is the calculation of $-2 - (-3)$.



You probably know a "rule" that does this exact thing. We'll talk about that later. For now, focus on the ideas and the concepts.

Try it: Calculate $1 - (-3)$ using an integer chip diagram. Then draw a number line picture to represent the calculation as movement on the number line.

4 In the end, the goal is not for you to have to actually count chips. The chips are symbols that help us to think through situations and understand the mathematical ideas behind the calculations. We can algebraically represent the process by writing the symbols without drawing the pictures.

$21 - 38$	21 positive chips minus 38 positive chips
$= 21 + (-38)$	21 positive chips combined with 38 negative chips
$= -17$	Create 21 zero pairs, leaving 17 negative chips

Note that the text here are thoughts you should have in your head, not necessarily words that you should be writing down. When writing up these problems, you should think the words even if you don't actually write them down.

Try it: Calculate $33 - 57$ by rewriting it as an addition calculation that could be done with integer chips and then perform the calculation.

28.1 Integer Chips - Worksheet 1

1 Calculate $3 + 4$ using an integer chip diagram and a number line diagram.

2 Calculate $-2 + (-5)$ using an integer chip diagram and a number line diagram.

3 Calculate $-2 + 6$ using an integer chip diagram and a number line diagram.

4 Calculate $3 + (-5)$ using an integer chip diagram and a number line diagram.

28.2 Integer Chips - Worksheet 2

1

Calculate $2-(-5)$ using an integer chip diagram and a number line diagram.

2

Calculate $-4-(-1)$ using an integer chip diagram and a number line diagram.

3

Calculate $-2-4$ using an integer chip diagram and a number line diagram.

4

Calculate $5-3$ using an integer chip diagram and a number line diagram.

28.3 Integer Chips - Worksheet 3

1 Perform the following calculations using a mental framework of integer chips.

This just means to think about the integer chip diagrams without actually drawing them.

$3 + 4 =$	$-2 + 8 =$	$-5 + 1 =$
$2 + (-5) =$	$8 + (-4) =$	$-3 + (-5) =$
$-4 + 6 =$	$-7 + 7 =$	$6 + (-4) =$

2 Perform the following calculations by rewriting them as addition calculations that can be done with integer chips and then performing the calculation.

Visualize or verbalize the process as you go through it, but you do not need to write words or draw pictures. Focus on which chips you start with, the chips you have after you've flipped some of them, and the cancellation of zero pairs.

$8 - 4 =$	$3 - 7 =$	$-2 - 4 =$
$4 - 9 =$	$-3 - 3 =$	$7 - 3 =$
$-5 - 3 =$	$5 - 7 =$	$5 - 2 =$

3 Practice your mental arithmetic by performing the following calculations.

$21 - 48 =$	$33 + (-15) =$	$-27 + 59 =$
$-32 + 15 =$	$29 - (-55) =$	$-18 + (-25) =$
$11 + (-36) =$	$-42 - (-26) =$	$-42 + 37 =$

4 Practice your mental arithmetic by performing the following calculations.

$218 - 134 =$	$-143 + 237 =$	$-115 + (-283) =$
$142 - 280 =$	$-248 - (-333) =$	$272 + (-358) =$

28.4 Integer Chips - Worksheet 4

1 There is a common phrase that is associated with subtraction: “Subtraction is addition of the opposite.” Integer chips and the number line can help to illuminate this concept. Consider the following diagram:

$$1 \quad + \quad (-1) \quad = \quad 0$$

How would you describe the idea that 1 and -1 are “opposite” numbers? How well does this idea extend to other numbers?

What does it mean for one thing to be the “opposite” of another thing?

2 Describe how integer chips can be used to demonstrate the idea that “subtraction is addition of the opposite.” (Hint: You may want to think about the physical manipulation of the chips when doing subtraction, and the relationship between a number and its opposite.)

3 The more technical term of the “opposite” of a number is the “additive inverse” of a number. The additive inverse of a number a is the number b that has the property that $a + b = 0$. Using this definition, we can see that -1 is the opposite of 1 since $1 + (-1) = 0$. Use this definition to argue that the additive inverse of the additive inverse of a number is itself.

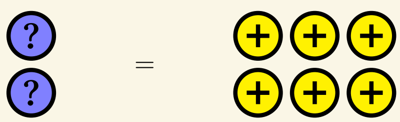
This is another tricky problem. Focus on the defining property of the additive inverse of a number.

28.5 Integer Chips - Worksheet 5

1 Integer chips can be combined with a variable chip to represent simple algebraic equations. Here is a representation of the equation $x + 2 = 6$:

The equation can be "solved" by adding the same type and number of chips to both sides of the equation until the unknown chip has been isolated. Describe the step required to isolate the unknown and determine its value.

2 The concept of division is making even groupings of things. The following is a visual demonstration of the fact that $2x = 6$ means that $x = 3$:



Draw a series of integer chip diagrams that demonstrate solving the equation $2x + 4 = -8$.

3 This type of visual representation of solving equations can be very helpful for introducing young children to algebraic reasoning. However, the practical use of this turns out to be very limited. Why do you think that is?

28.6 Deliberate Practice: Practice with Negatives

Focus on these skills:

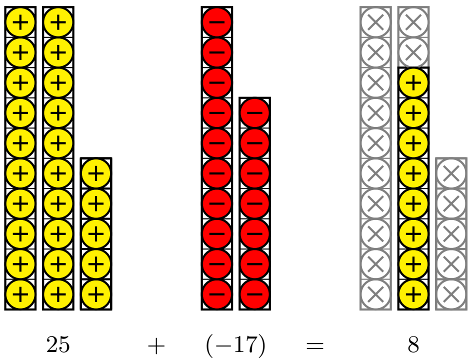
- Mentally visualize collections of integer chips to help think through the calculation.
- Present your work legibly.

Instructions: Perform the given calculation mentally.

1. $24 + (-38)$
2. $-36 - 57$
3. $25 - (-18)$
4. $-23 + 74$
5. $-44 - (-57)$
6. $35 + (-68)$
7. $-29 - 55$
8. $-242 + (-159)$
9. $186 - (-372)$
10. $-218 - (-386)$

28.7 Closing Ideas

In this section, we introduced the idea of integer chips, and used them to describe all integer addition and subtraction problems, using both positive and negative integers. But we're not limited to just these things. We can take the idea of these chips and use them with the idea of base-10 blocks by replacing the cubes with chips.



We can also imagine breaking each of the chips into wedges and using them for addition and subtraction of fractions. And since decimals are just fractions, we can also imagine using them with decimals. Not that we would actually do that, but this is is conceptually possible.

Unless the chips were fairly large to begin with, this would be a total mess to track in reality!

We also saw in the last couple worksheets that we can connect these chips to talking about the opposite of numbers and solving equations.

This flexibility is one of the marks of a beautiful idea in mathematics. The ability to take one idea and connect it to so many other ideas means that we now have an expanded vocabulary with which to talk about our mathematical ideas. It also means that when it comes down to solving problems, we have yet another tool in our toolbox. And broadening that toolbox is the ultimate goal of college mathematics.

28.8 Going Deeper: Mental Arithmetic

For many students, mental arithmetic usually feels quite complicated and can be extremely frustrating. One of the big challenges for mental arithmetic is that most people try to do mental arithmetic the way they do arithmetic on paper. Unfortunately, this is both extremely inefficient and unnecessarily difficult. The primary reason has to do with how the paper method relies on being able to write down a digit to remember it later, but our brains simply cannot do that.

To get a sense of just how central that memory is, we're going to work through adding three three-digit numbers together, and we're going to put a box around every number that appears in the process:

$$\begin{array}{r} 387 \\ 236 \\ +523 \\ \hline \end{array}$$

- Ones digit: $\boxed{7}$ plus $\boxed{6}$ is $\boxed{13}$, plus $\boxed{3}$ more is $\boxed{16}$. Write the $\boxed{6}$ below and carry the $\boxed{1}$.
- Tens digit: $\boxed{1}$ plus $\boxed{8}$ is $\boxed{9}$, plus $\boxed{3}$ more is $\boxed{12}$, plus $\boxed{2}$ more is $\boxed{14}$. Write the $\boxed{4}$ below and carry the $\boxed{1}$.
- Hundreds digit: $\boxed{1}$ plus $\boxed{3}$ is $\boxed{4}$, plus $\boxed{2}$ more is $\boxed{6}$, plus $\boxed{5}$ more is $\boxed{11}$. Write down these digits to get the final result.

Let's make a list of all the digits that appeared in the process, keeping track of the order in which they appeared:

$$\boxed{7}\boxed{6}\boxed{13}\boxed{3}\boxed{16}\boxed{6}\boxed{1}\boxed{1}\boxed{8}\boxed{9}\boxed{3}\boxed{12}\boxed{2}\boxed{14}\boxed{4}\boxed{1}\boxed{1}\boxed{3}\boxed{4}\boxed{2}\boxed{6}\boxed{5}\boxed{11}$$

When you look at this string of numbers, the critical observation to make is that the actual answer to the calculation does not appear anywhere. In fact, when you look at the numbers, you may have a hard time even finding the digits that make up the answer. We'll indicate those values with an arrow:

$$\begin{array}{ccccccccccccccccccccccc} \boxed{7}&\boxed{6}&\boxed{13}&\boxed{3}&\boxed{16}&\boxed{6}&\boxed{1}&\boxed{1}&\boxed{8}&\boxed{9}&\boxed{3}&\boxed{12}&\boxed{2}&\boxed{14}&\boxed{4}&\boxed{1}&\boxed{1}&\boxed{3}&\boxed{4}&\boxed{2}&\boxed{6}&\boxed{5}&\boxed{11} \\ &&&&&\uparrow&&&&&&&&&\uparrow&&&&&&&&\uparrow \end{array}$$

Now that they've been marked, notice how many numbers appear in between them. These digits are numbers that you're trying to hold in memory while all the other numbers are going through your head. Between the 6 for the ones digit and the 11 for the hundreds and thousands digit in the last step, there are 16 numbers in between, and one of those other numbers is one you were supposed to have memorized.

When following this algorithm on paper, our brains are allowed to forget those digits because they're written down. But when we do this mentally, we have to try to keep those numbers in memory while other digits are being processed. While it helps to have reached a level of automaticity so that you're less consciously doing those computations, it's still quite challenging for most people.

There is a technique of converting the digits into letters that you convert into a word to help remember them. The idea is that it's easier for your brain to hold a word in memory while it processes numbers rather than holding a number in memory while processing numbers. But that technique requires additional types of practice and memory training.

The standard addition algorithm uses a digit manipulation scheme. What this means is that we're looking at each digit as its own object. In the calculation, the number 387 is never used as a number. It's broken down into three separate pieces: 3, 8, and 7. This causes extra strain on our brains when handing it because it requires more brain power to think about three separate objects rather than thinking about one.

The big transition that makes mental arithmetic easier is to reorganize the calculation so that we don't have to remember numbers for as long. The representations of arithmetic on the number that we developed a few sections ago are a big towards that end. Here is how we're going to translate the process:

$$\begin{array}{r} 387 \\ 236 \\ \underline{+523} \end{array} \quad \longrightarrow \quad \begin{array}{l} 387 \\ +200+30+6 \\ +500+20+3= \end{array}$$

Here is how we're going to work through it:

- Add the first two numbers: $\boxed{387}$ plus $\boxed{200}$ is $\boxed{587}$, plus $\boxed{30}$ more is $\boxed{617}$, plus $\boxed{6}$ more is $\boxed{623}$.
- Add the next number: $\boxed{623}$ plus $\boxed{500}$ is $\boxed{1123}$, plus $\boxed{20}$ more is $\boxed{1143}$, plus $\boxed{3}$ more is $\boxed{1146}$.

This is how those numbers look written in order:

$$\boxed{387}\boxed{200}\boxed{587}\boxed{30}\boxed{617}\boxed{6}\boxed{623}\boxed{623}\boxed{500}\boxed{1123}\boxed{20}\boxed{1143}\boxed{3}\boxed{1146}$$

Let's make some observations about this

- The total number of boxed values is significantly lower. There were 23 in the first method and only 14 in the second method.
- However, the numbers that we were working with are also significantly larger. In the first method, we only needed to be comfortable with small addition calculations, whereas the second method requires a level of fluency with adding in different place values.
- The new approach does not require any long term memory, and the final number is the complete final answer. In fact, the process is basically just keeping a running total as you work your way through, so that you only really have to remember keep three numbers in your head at any time (the previous total, the number you're adding, and the new total).

As with most things mathematical, this should not be seen as a "rule" for how you're "supposed to" do mental arithmetic. It turns out that people who are good at mental arithmetic employ a number of different techniques based on patterns that they identify as being familiar. It's not necessary to always go in the same order every time, especially if there are patterns that make more intuitive sense.

For example, in this calculation, you might notice that $300 + 200 + 500 = 1000$. This is a number that you can put into memory (with a little bit of practice) because it's a "nice" number. And then you would only need to add three two-digit numbers together instead. You might also notice that $387 + 3 = 390$, which also brings you to a nice number. Then you can add in the hundreds and tens digits, leaving the final 6 for the very last step.

You could also use other groupings if they make sense. For example, you might see

$$387 + 236 = 387 + 13 + 223 = 400 + 223 = 523$$

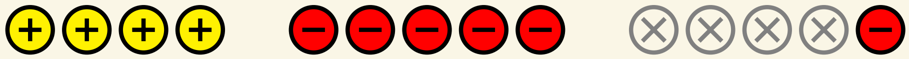
and do that faster than you would do

$$387 + 200 + 30 + 6 = 587 + 30 + 6 = 617 + 6 = 623.$$

The main point is that mathematics favors those who are flexible in their thinking instead of being rigid. Throughout this book, we've emphasized just how restraining a rule-based approach to mathematics can be, and this thread follows all the way back to the ways that you've learned your basic arithmetic. Hopefully, as you've been working your way through this book, you've started to move away from those things and have started to build a deeper understanding of mathematics.

28.9 Solutions to the "Try It" Examples

1

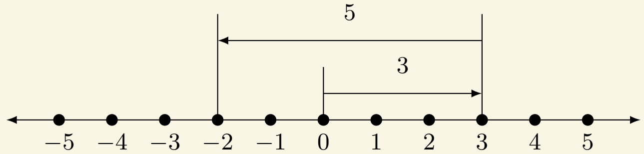


$4 \quad + \quad (-5) \quad = \quad -1$

2

$3 \quad + \quad -5 \quad = \quad -2$

3 steps right | 5 steps left | 2 steps left



$\boxed{3 + (-5) = -2}$

3

$1 \quad - \quad (-3)$

$= \quad 1 \quad + \quad 3 \quad = \quad 4$

4

$$\begin{aligned} 33 - 57 &= 33 + (-57) \\ &= -24 \end{aligned}$$

Section 29

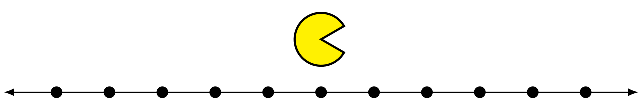
Turn Around and Walk Backwards: Arithmetic as Movement on a Number Line

Learning Objectives:

- ☐ Represent numbers as movement along the number line.
- ☐ Interpret addition and subtraction of numbers as movements along the number line.
- ☐ Understand the equivalence of different types of movements.

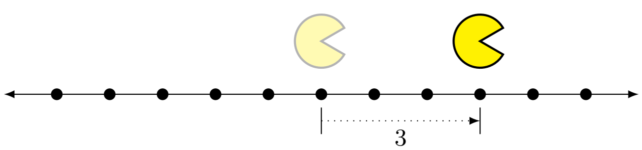
In the previous sections, we have discussed the idea of both numbers and arithmetic being represented by movement. We're going to formalize this idea a bit more in this section.

We will begin by thinking about numbers as arrows that indicate movement. When we were looking at integer chips, we talked about taking a certain number of steps to the left or to the right. It is actually more useful to think about this as taking a certain number of steps forward or backward. Imagine that you are standing on the number line facing to the right:

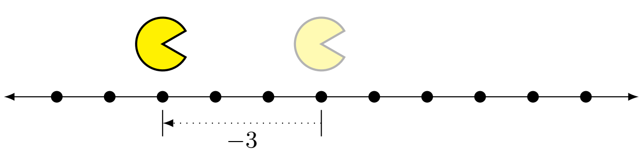


We have placed dots but no numbers to emphasize the idea that this is about movement and not location.

Every number represents movement. Positive numbers represent forward steps, negative numbers represent backwards steps, and zero represents taking no steps. It is important to focus on the idea that this is an instruction about movement, not location. This means that 3 means 3 steps forward no matter where you are on the number line.

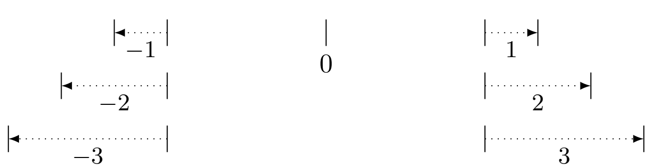


Similarly, -3 means 3 steps backwards no matter where you are on the number line.



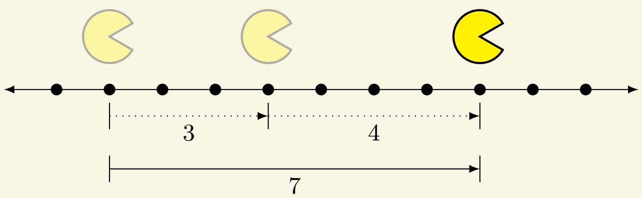
It is very important that the orientation remain consistent. The default direction is to face to the right, so that the default for "forward" is to the right. From this idea, we can actually reduce the entire picture to just arrows (though we'll stick with including the figure for emphasis).

These arrows are connected to the mathematical concept of vectors. We'll touch on them briefly in the worksheets, but this an advanced topic that goes beyond the scope of this book.



1

Addition of two numbers means to perform two sets of movements in sequence. So $3 + 4$ means to move 3 steps forward followed by 4 steps forward. The end result is the same as 7 steps forward, so $3 + 4 = 7$.



$3 + 4 = 7$

Try it: Draw a movement diagram to represent $2 + 4$ and compute the result.

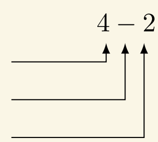
We will use a dotted line for the intermediate steps and a solid line for the final result.

2

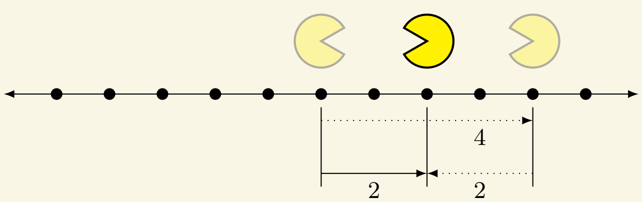
Subtraction is the same idea, except that we face to the left before moving for subtracted values. This fits in with the concept of subtraction undoing addition, since if you take a certain number of steps forward then turn around and do it again, you will end up right back where you started.

When reading a calculation, you want to be able to read it as a series of steps from left to right.

Step 0: Always start facing right
Step 1: Four steps forward
Step 2: Subtraction means to face left
Step 3: Two steps forward



In the diagram, it is important to draw the figure facing the correct direction to really emphasize the point.



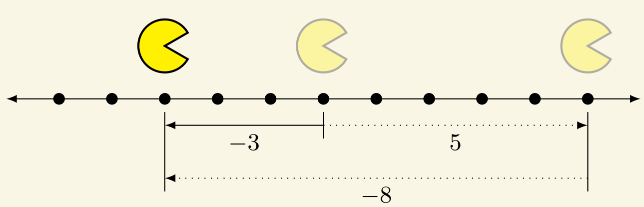
$4 - 2 = 2$

Try it: Draw a movement diagram to represent $2 - 5$ and compute the result.

You may be facing the opposite direction, but it's still the same as having taken zero steps.

3

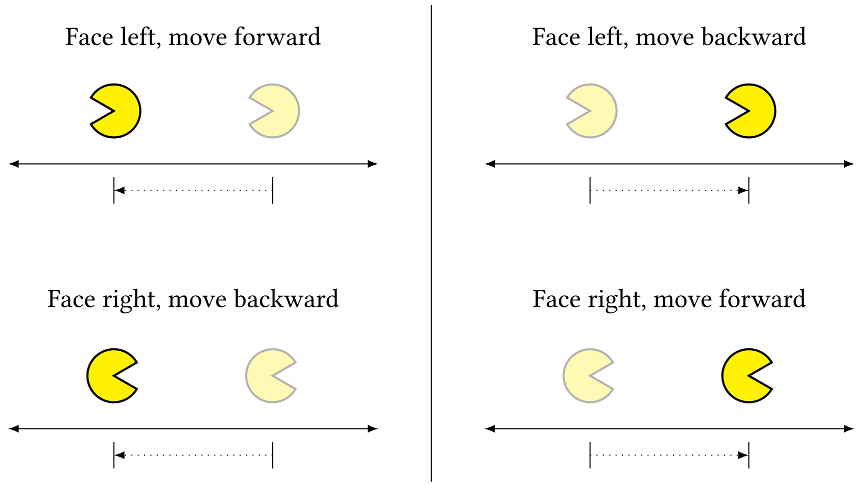
If we replace the values with negative numbers, the only thing that changes is that instead of forward steps we use backward steps. This does not change the direction that we're facing.



$5 + (-8) = -3$

Try it: Draw a movement diagram to represent $3 - (-4)$ and compute the result.

With this representation in place, we can very quickly understand and visualize a particular relationship between addition and subtraction. Notice that facing right and walking backward results in the same type of movement as facing left and walking forward. Similarly, facing right and walking forward is the same as facing left and walking backward.



These diagrams show us the following relationships:

(Top-left)	$a - b = a + (-b)$	*(Bottom-left)*
(Top-right)	$a - (-b) = a + b$	*(Bottom-right)*

Make sure you study the pictures and equations closely!

These ideas give us a clear representation of "subtraction is addition of the opposite." Walking forward (no matter which way you're facing) gives you the same result as turning around and walking backward. At a very deep level, this is just another way to see that addition and subtraction are deeply related concepts.

We saw earlier that addition has properties like commutativity and associativity, so we actually view addition as the fundamental operation, and subtraction is the inverse of addition.

29.1 Number Line Movement - Worksheet 1

1 Draw a movement diagram to represent $2 + 4$ and compute the result.

Make sure that you are facing the correct direction!

2 Draw a movement diagram to represent $3 - 7$ and compute the result.

3 Draw a movement diagram to represent $-4 + 5$ and compute the result.

4 Draw a movement diagram to represent $-3 - 2$ and compute the result.

29.2 Number Line Movement - Worksheet 2

1 Draw a movement diagram to represent $4 + (-3)$ and compute the result.

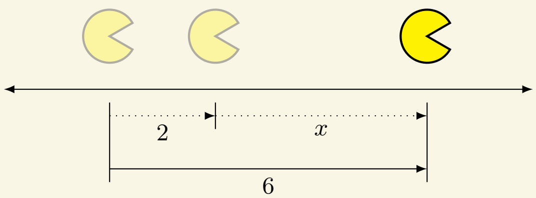
2 Draw a movement diagram to represent $2 - (-4)$ and compute the result.

3 Draw a movement diagram to represent $-1 + (-3)$ and compute the result.

4 Draw a movement diagram to represent $-4 - (-7)$ and compute the result.

29.3 Number Line Movement - Worksheet 3

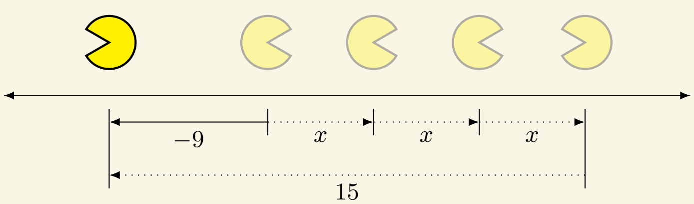
1 We have seen that the visualizations that we've created give us another way to look at solving equations. For example, we can use our movement diagram to visualize the equation $2 + x = 6$:



Convert the algebraic equation $2 + x = 6$ into a question about the diagram above. Be sure that your words include enough information that the diagram can be reconstructed by analyzing the words.

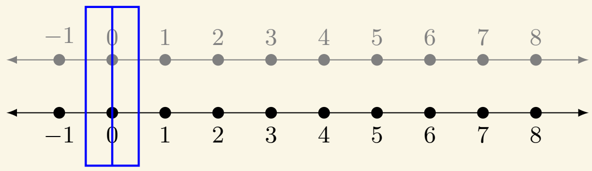
You don't need an example. Think about how you might explain this diagram to a middle school student.

2 Convert the diagram into an algebraic equation, then solve it.

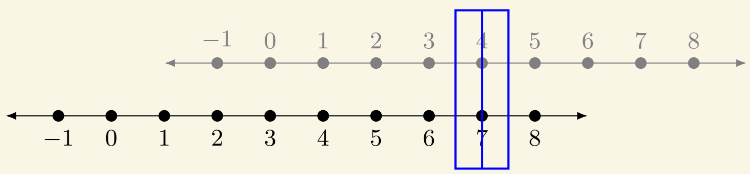


29.4 Number Line Movement - Worksheet 4

1 We can replace the idea of movement with stacked number lines and a slider.



You can think of this as two rulers. To represent $a + b$, slide the top ruler over by a spaces, then move the slider to the position b on the upper ruler. The value on the bottom ruler marked by the slider gives the result. Here is what $3 + 4$ would look like.



- Step 1: Slide the top ruler 3 spaces to the right. Notice that gray 0 lines up with the black 3.
- Step 2: Move the blue slider over the top of the gray 4.
- Step 3: Read the final answer from the bottom line.

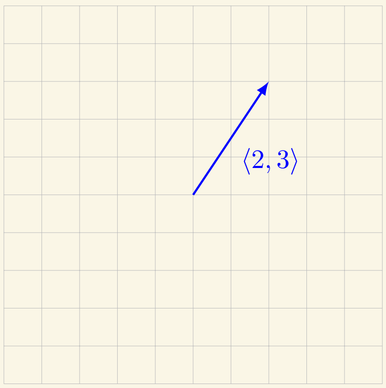
Draw a stacked number line diagram to calculate $3 + (-2)$.

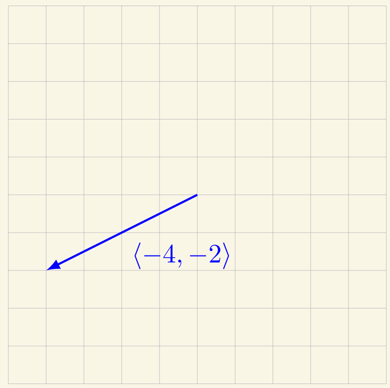
Although this may seem like an artificial construction, this idea basically mimics the way that calculations were done with slide rules. Unfortunately, there is one piece missing, which is the application of logarithms. And logarithms go beyond the scope of this course. But one of the core features of logarithms is that it converts multiplication into addition. And that is the trick that makes slide rules work.

Slide rules have been replaced by calculators, but it's of historical interest that slide rules were the calculators of their day, which spanned from the early 1600s through the mid-1900s. Slide rules were phased out during the 1970s when pocket calculators started to become cheap enough for the average person to be able to purchase one.

29.5 Number Line Movement - Worksheet 5

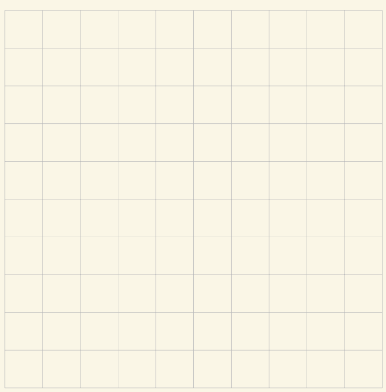
1 We've considered arithmetic as movement on the number line. But it turns out that we can do the same thing on the coordinate plane. But rather than using a single number, we use a pair of numbers to indicate motion in the horizontal and vertical directions. Here are a couple different vectors.





Notice the use of pointed brackets instead of round brackets. This is how we distinguish points from vectors.

In the same way that addition with a movement diagram was simply doing one movement followed by another one, addition of vectors can be thought of as doing one movement followed by another one. With this concept in mind, sketch a picture of $\langle 2,3\rangle + \langle -4,-2\rangle$ on the grid and then compute the result.



Remember that the result is the final result of all the movement. What would the arrow look like if you simply went to the final position directly?

2 Based on the ideas in this section, what do you think $-\langle 1,5\rangle$ means? How would you justify or explain your idea?

Hint: What is the relationship between 3 and -3?

29.6 Deliberate Practice: Movement on the Number Line

Focus on these skills:

- Indicate the direction that you are facing at each step.
- Indicate the motion at each step using arrows.
- Present your work legibly.

Instructions: Draw a movement diagram to represent the calculation and compute the result.

1. $3 + 2$
2. $2 - 5$
3. $6 - 4$
4. $-3 + 5$
5. $-5 - (-3)$
6. $-4 + (-2)$
7. $3 + (-4)$
8. $4 - 9$
9. $-3 + 7$
10. $-4 - (-3)$

29.7 Closing Ideas

We have reached the end of the exploration of addition and subtraction. We have covered a variety of ways of thinking about those mathematical operations. Remember that the point is not that you need to use all of these ideas all of the time, but rather that having that diverse toolbox of ideas and approaches will give you more flexibility to solve problem you come across in the future.

Many students think of math as a set of rules that you need to follow, rather than a diverse set of concepts that are related to each other. That conceptual diversity in many ways reflects the diversity of students. Different students come to mathematics with different experiences and intuitions. Perhaps one student finds the movement diagrams easier to understand while another student thinks that base-10 blocks are easiest. It doesn't really matter that much. The goal is for students to come to their own experience of understanding of mathematical thinking and the diversity of approaches in the last several sections helps to reflect the diversity of perspectives that exist within mathematical reasoning.

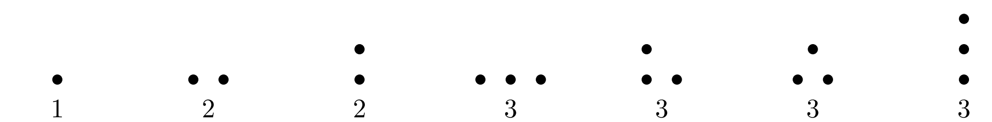
We'll close this section with a very important note. If you find one way of looking at something to be difficult to understand, that's usually a signal to probe more deeply into it. It's not a signal that you should simply do something else. It is far better to take an intellectual posture that seeks to understand rather than seeks to not understand. Having a broader understanding of ideas only stands to benefit you. Maybe you will eventually understand it, or maybe not. But if you don't try, you definitely won't.

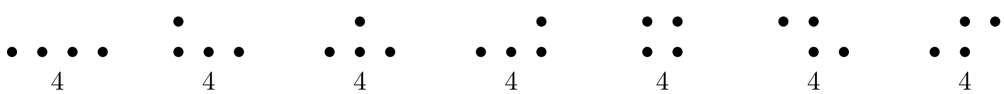
29.8 Going Deeper: Number Patterns

At this point, we have a wide range of tools for representing numbers and arithmetic. We've used movement on the number line, base-10 blocks, and integer chips to help build out our understanding of addition and subtraction. But it turns out that this is just the beginning.

What we have been discussing is a way to represent numbers. In particular, we've been representing numbers in the service of thinking through arithmetic. There's a completely different way to represent numbers where we can begin to understand numbers on their own terms and make discoveries about the numbers themselves.

We're going to strip everything back to one of the most primitive representations of numbers that we have. We can think of numbers as the quantity represented by a collection of objects. We will use dots as those objects. A key idea here is that the organization of those dots is not what defines the number, but simply the quantity of dots. This means that we can have multiple representations of the same number. Here are some examples:



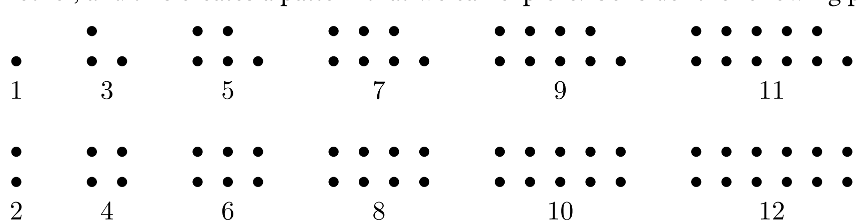


Fans of a certain Russian game from the 1980s will appreciate this row.

There is already so much that can be said about these dot arrangements, and we've only gone up through 4.

- All but one of the shapes is built on a square grid pattern. That square pattern is helpful for highlighting the ideas we'll be discussing later, but is not necessary to have that in a representation. For example, there's a triangle for 3 that isn't set up that way.
- In the first row, some of the patterns are just rotations of each other. In some ways, they might be seen as the same, but there are some contexts in which we care about whether the dots are in a row or in a column.
- In the bottom row, none of the figures are rotations of each other, but some are reflections of each other. This is another type of relationship that we can set up with numbers.

As we create more representations of numbers, we can find that some numbers share features with each other, and this creates a pattern that we can explore. Consider the following patterns:



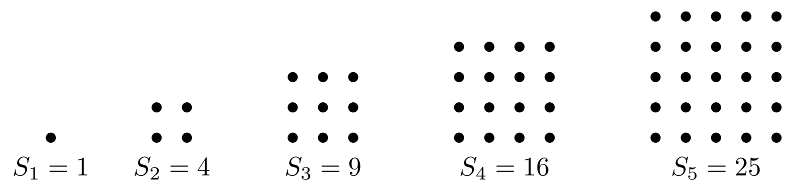
This pattern is highlighting a basic feature of numbers, which is that some of them are even and some of them are odd. This is something inherent to the numbers themselves. In some sense, we do not make the even numbers even and the odd numbers odd. It's just the case that some numbers can be broken into two equal-sized groups, and that others can't. And the ones that can't will always have one extra object by itself. This seems to be an intrinsic property of the numbers.

The extra dot is the "odd man out."

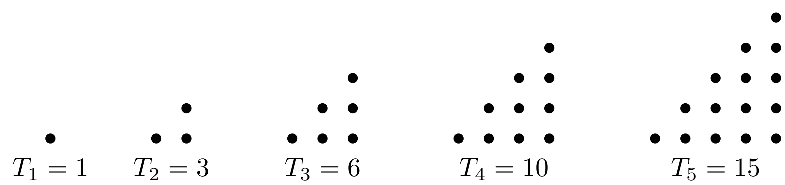
This isn't the only place that these types of number patterns show up. Here are a few more examples.

- Square Numbers: S_n is the nth square that we can make with dots.

Think about the meaning of this relative to the algebraic concept of squaring.

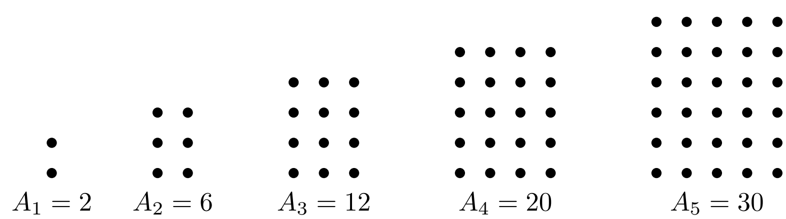


- Triangular Numbers: T_n is the nth triangle of that we can make with dots.



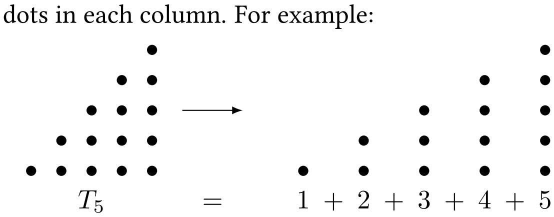
- Almost-Square Numbers: A_n is the nth rectangle that is either one column of dots short or being a perfect square.

This is not an "official" math terminology. Sometimes, mathematicians make up names for things just so that they can reference them, even if nobody else knows what they're talking about.



As we construct these geometric patterns, it's helpful to come up with algebraic expressions that can represent them. The square numbers are somewhat obvious. The nth square number is just n^2, which we can write as $S_n = n^2$. The almost-square numbers are a little bit trickier, but we can look at the pictures and see that there's always one more row than the number of columns, and so the nth almost-square number is $n(n+1)$. We can write this as $A_n = n(n+1)$.

But the triangular numbers are trickier. One way we can express those numbers is by writing them as a sum of the dots in each column. For example:



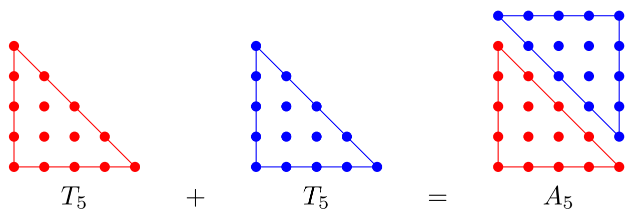
We can write this more generally as

The $\cdots$ means that you continue the pattern until you reach the end.

$$T_n = 1 + 2 + 3 + \cdots + n.$$

This gives us a computational method for calculating triangular numbers. One drawback to this is that if we wanted to find a very large triangular number, such as the one millionth one, we would end up having to add up a million numbers. Mathematicians see these situations as a challenge. Is there a better way to calculate triangular numbers?

It turns out that there is, but it's a little bit tricky. And this is where a lot of creative mathematical thinking starts to come into play. Look at the following diagram:



If you try drawing a few other pictures, you'll see that this pattern persists. This means that we have the relationship $2T_n = A_n$, which can be written as $T_n = \frac{A_n}{2}$. But we established earlier that $A_n = n(n+1)$, and so we must have $T_n = \frac{n(n+1)}{2}$. And this gives us a different formula for the triangular numbers, but now we're able to easily compute the millionth triangular number.

Mathematicians call this a *closed form formula*. It basically means that we can calculate values directly from the formula without needing to do excessive calculations.

But now we can also combine our two formulas for triangular numbers and get an even more interesting result:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

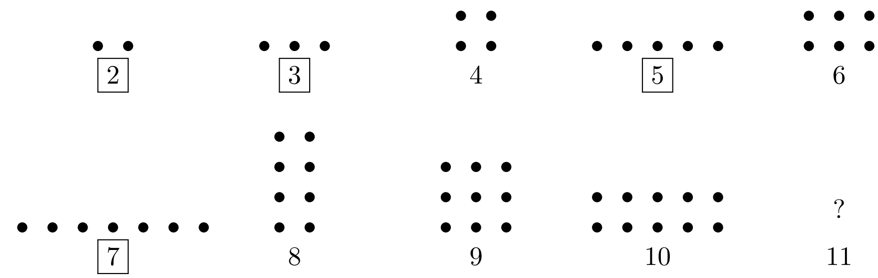
This gives us a somewhat surprising formula for adding up consecutive numbers starting from 1. And we found it as the result of exploring patterns of numbers by combining both an algebraic and geometric perspective.

This line of thinking can lead us to explore other possibilities. Instead of adding every number, what happens if we only add the odd numbers together? For example, we can see that $1+3+5=9$ and $1+3+5+7=16$. Do those numbers look familiar? This hints at a mathematical relationship between the odd numbers and perfect squares. We're not going to elaborate any further on that relationship, but here is a diagram to contemplate as you think about why this relationship seems to exist.

See if you can understand the pattern and describe it in words. Then think about how you can be confident that this pattern will continue. This exercise is simple, but it's not easy.

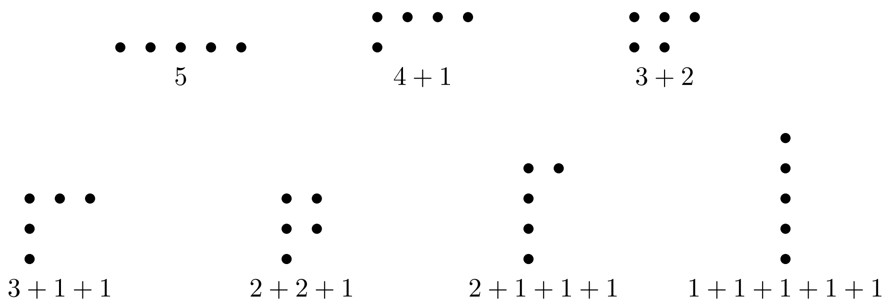
There are other investigations that can be launched by exploring these patterns of dots.

- Some numbers can form rectangles, but others can only be represented as a string of dots in a single line. Is there anything special about those numbers? How many of these numbers are there?



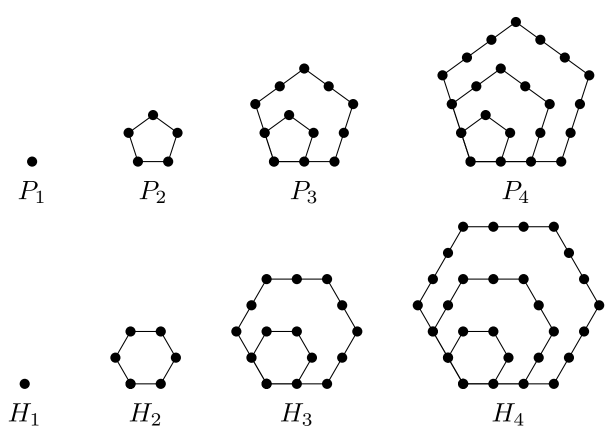
- Dots can be used to represent ways of writing numbers as the sums of numbers less than or equal to them. We can look at this as adding the individual rows. Is there a way to determine how many ways we can break up a number into pieces like this?

Or maybe we should add columns. Or perhaps it doesn't even matter?



- We can reject the underlying rectangular grid and look at other patterns involving shapes of different numbers of sides. What patterns can be found here?

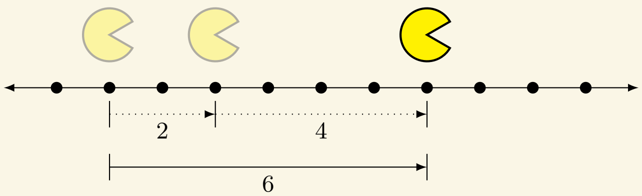
P stands for pentagon and H stands for hexagon.



There is an endless supply of questions like these. The exploration of these number patterns is at the heart of an area of mathematics known as *number theory*, which is one of the oldest topics of mathematical study. We'll have to leave the topic here, but hopefully this brief introduction will encourage you to explore some new ways of thinking about mathematics.

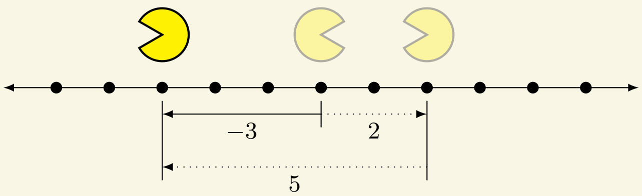
29.9 Solutions to the "Try It" Examples

1



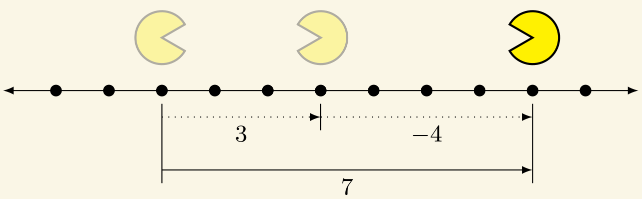
$\boxed{2 + 4 = 6}$

2



$\boxed{2 - 5 = -3}$

3



$\boxed{3 - (-4) = 7}$

Making Groups of Things: Multiplication

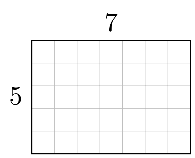
Learning Objectives:

- ☐ Understand multiplication as A groups of B.
- ☐ Understand multiplication as areas.
- ☐ Understand multiplication with negatives as creating “opposite” groups.

In this book, we’ve already seen two different concepts for multiplication. When we were working with the distributive property, we talked about multiplication representing groups of things and used this example:

$$4 \cdot (2 \text{ apples} + 3 \text{ oranges}) = 8 \text{ apples} + 12 \text{ oranges}$$

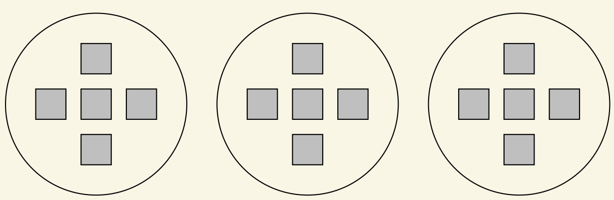
We also talked about multiplication being represented as an area:



$\longrightarrow \quad 5 \cdot 7 = 35$

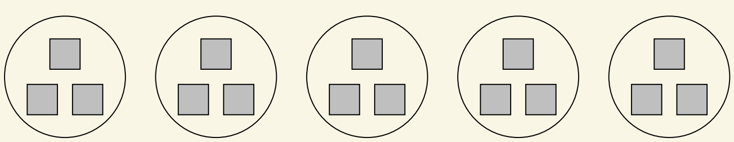
In this section, we’re going to explore these multiplication concepts more deeply in order to understand how they relate to each other and to some of our other representations of numbers.

1 The concept of using groups to represent multiplication is often referred to as “A groups of B” (meaning the number of objects you have when you have A groups of B objects each). If we think about 3 groups of 5, we can represent those as individual objects in the following manner:



$\rightarrow 3 \cdot 5 = 15$

It’s important to recognize that this is not the same picture as 5 groups of 3, even though the total number of objects is the same.



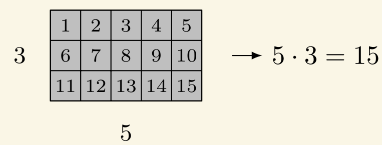
$\rightarrow 5 \cdot 3 = 15$

Try it: Draw a diagram to represent $2 \cdot 4$ and $4 \cdot 2$.

Be sure to make it clear which one is which!

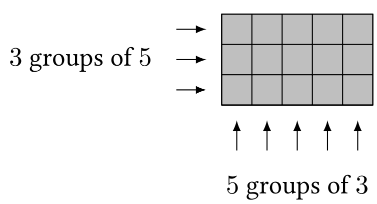
2 The concept of area as a representation of multiplication comes from the formula for the area of a rectangle: $A = \ell \cdot w$ (where ℓ is the length of the rectangle and w is the width). The area of a shape is the number of unit squares that we can fit into it. In the case of rectangles, things fit perfectly (compared with other shapes, like circles, where you have lots of pieces of squares to deal with).

There is no formal convention for whether the length is the horizontal direction or the vertical direction.



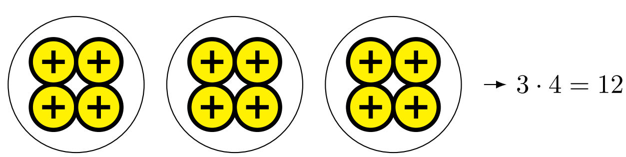
Try it: Represent $4 \cdot 7$ using a rectangle and determine the product by determining the area.

We can start to see the connection between thinking about groups and thinking about areas if we simply think about the groups being either horizontal or vertical collections of squares.

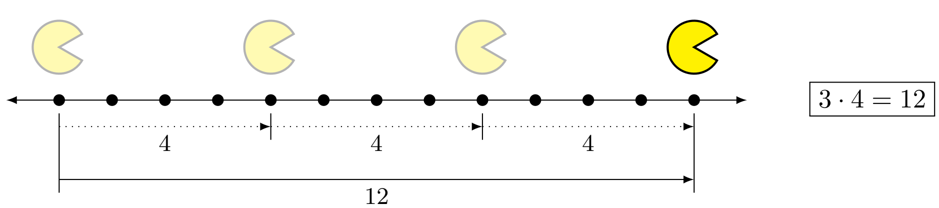


These diagrams work well for positive numbers, but if we want to use negative numbers, we need a more robust image. We can look at this either with integer chips or with movement diagrams. We will work our way through the different possibilities.

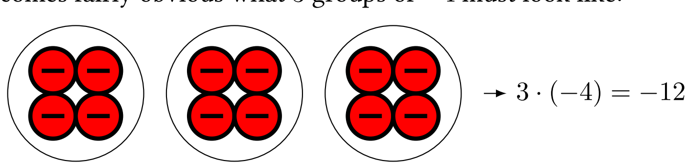
To do positive groups of positive objects, it's not significantly different than what we've already done. Here are 3 groups of 4 integer chips:

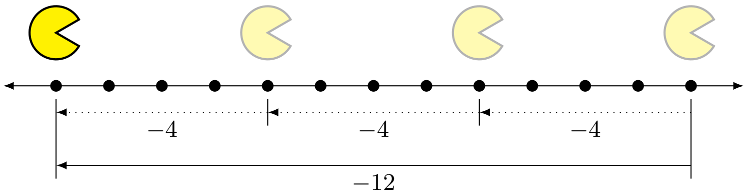


And here are 3 movements of 4:



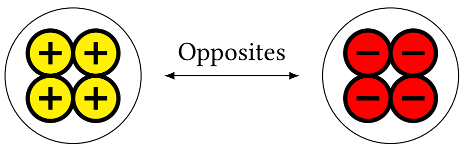
From here, it becomes fairly obvious what 3 groups of -4 must look like:



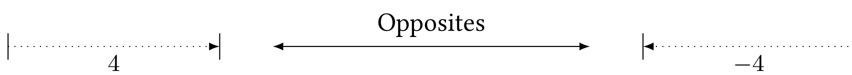


$\boxed{3 \cdot (-4) = -12}$

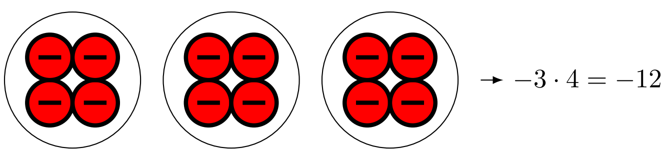
But this leads us to try to figure out what a negative number of groups might mean. Instead of using the word “negative” it is better to think of this as “opposite.” If we have a group of integer chips, it is pretty clear what the corresponding “opposite” group would be:



Similarly, given a particular movement, the opposite is the same total movement but in the opposite direction.

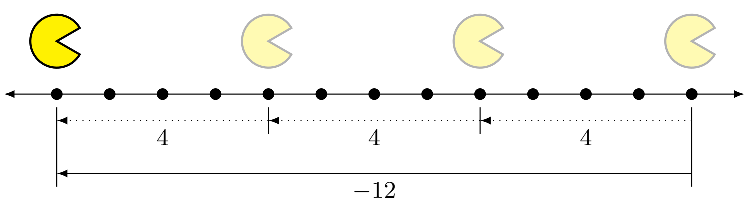


With this in mind, the product $(-3) \cdot 4$ would be three groups of the “opposite” of 4. in other words, three groups of four negative integer chips.



3 groups of the “opposite” of 4

And also three movements that are the “opposite” of 4:



$\boxed{-3 \cdot 4 = -12}$

3 groups of the “opposite” of a movement of 4

3

Try it: Using the idea of “opposite” groups and “opposite” movements, determine the value of $-3 \cdot (-4)$ using both an integer chip diagram and a movement diagram.

We can condense this all into a familiar set of “rules” (but remember that it’s not about the rule, but about the logic that gives us the rule).

Theorem 30.1. When multiplying numbers, the sign of the final result behaves according to the following:

- (positive) · (positive) = (positive)
- (positive) · (negative) = (negative)
- (negative) · (positive) = (negative)
- (negative) · (negative) = (positive)

30.1 Multiplication - Worksheet 1

1 Calculate $4 \cdot 6$ using A groups of B and an integer chip diagram.

2 Calculate $-3 \cdot 5$ using A groups of B and an integer chip diagram.

3 Calculate $5 \cdot (-2)$ using A groups of B and an integer chip diagram.

4 Calculate $-4 \cdot (-5)$ using A groups of B and an integer chip diagram.

30.2 Multiplication - Worksheet 2

1 Calculate $3 \cdot 6$ using A groups of B and a movement diagram.

2 Calculate $4 \cdot (-2)$ using A groups of B and a movement diagram.

Which way should you be facing?

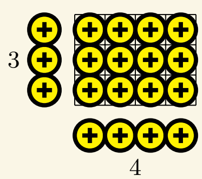
3 Calculate $-3 \cdot 5$ using A groups of B and a movement diagram.

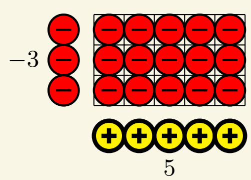
4 Calculate $-5 \cdot (-4)$ using A groups of B and a movement diagram.

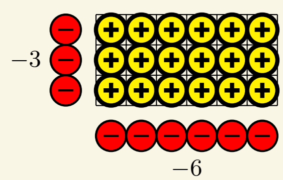
30.3 Multiplication - Worksheet 3

1 Represent $5 \cdot 6$ using a rectangle and determine the product by determining the area.

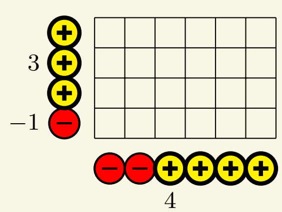
2 The concept of counting up the area doesn't really make sense if we start to involve negative numbers. However, we can mix the integer chips with the basic multiplication rules in order to create pictures with "signed" areas. What this means is that each block of area counts as either a positive or negative. You only need to maintain the basic rule of signs for each box. Here are some examples.





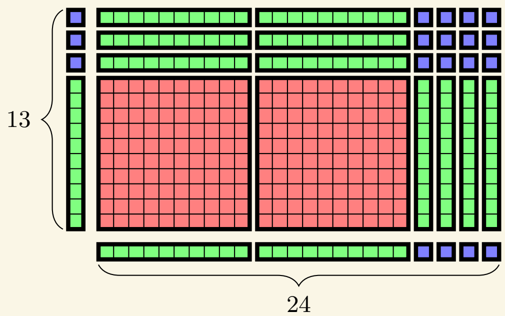


Something that's interesting about this approach is that is remains consistent even if we more complex expressions. Use the grid below to draw a signed area diagram for the the product $(3 + (-1)) \cdot (-2 + 4))$. Calculate the result by counting the values in the grid and then by directly simplifying the expression.



30.4 Multiplication - Worksheet 4

1 In the same way that we used base-10 blocks to help us to organize information for addition and subtraction, we can also use them for organizing information for multiplication. When working with larger numbers, we can use organize the calculation of areas using the base-10 blocks to measure the side lengths instead of using individual squares.



Use the diagram to determine the product $24 \cdot 13$.

You should be able to count up the value mentally by working through the diagram in an organized manner.

2 Draw a diagram with base-10 blocks to calculate $14 \cdot 32$.

3 Visualize a diagram with base-10 blocks to calculate $14 \cdot 12$.

Stay organized by thinking of an organized way to think through the diagram.

30.5 Multiplication - Worksheet 5

1 We are going to analyze the “standard” multiplication algorithm. There are actually several multiplication algorithms, and it’s not necessarily the case that the way it’s presented here is the way you learned it. But this is the most common way it’s taught in the United States.

There is also the lattice algorithm, the “peasant” algorithm, and the box/grid method, to name a few.

Most people who can do this calculation have a difficult time describing it in terms other than the specific steps. (For example, multiply this by that, then this by that, then make sure you write a zero...) But we want to take the time to actually understand why these steps are what they are. To help, we are going to put the standard algorithm side-by-side with the grid method.

Remember that we used the grid method when multiplying polynomials and that it’s reflecting the idea that multiplication is an area.

$$\begin{array}{r} \scriptstyle 1\;\; \\ \scriptstyle 5\;\; \\ 3\,7 \\ \times\, 2\,8 \\ \hline 2\,9\,6 \\ +\,7\,4\,0 \\ \hline 1\,0\,3\,6 \end{array}$$

	30	7
20	600	140
8	240	56

If you are unfamiliar with this multiplication algorithm, you may want to find an online video that describes it.

In performing the standard algorithm, you compute four separate multiplication calculations: 7×8, 3×8 (don’t forget the carried terms), 7×2, and 3×2 (again, don’t forget the carried terms). Explain how the four boxes in the grid method correspond to the four products in the standard algorithm. Be sure to explain the roles of the zeros after the numbers in the grid method compared to the standard algorithm.

2 In the “middle” portion of the standard algorithm, we come across the numbers 296 and 740. Those numbers do not directly appear in the grid method, but those values do correspond to a certain aspect of the grid method. Explain how you can get the numbers 296 and 740 from the grid method.

3 The last step of the standard algorithm is to add the two values from the “middle” portion. The last step of the grid method is to add the values in the boxes together. Verify that you get the same result using both methods.

30.6 Deliberate Practice: Multiplication Concepts

Focus on these skills:

- Draw the appropriate diagrams based on the instructions.
- Present your work legibly.

Instructions: Perform the indicated calculation using A groups of B and a movement diagram.

1 $3 \cdot (-4)$

2 $(-4) \cdot 3$

3 $2 \cdot (-5)$

4 $3 \cdot 2$

5 $(-2) \cdot (-4)$

6 $(-3) \cdot 5$

Instructions: Perform the indicated calculation using a diagram of base-10 blocks.

7 $12 \cdot 24$

8 $24 \cdot 16$

9 $11 \cdot 32$

10 $26 \cdot 14$

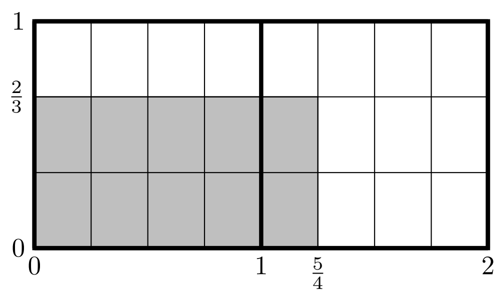
30.7 Closing Ideas

One way of understanding multiplication that was not discussed here was "multiplication is repeated addition." We side-stepped that one because it's a very limiting perspective of multiplication. The limitation of repeated addition is that it is limited to only integer values. It does not make sense to add something a half of a time. But it does make sense to talk about half groups of objects, such as a half bags of chips.

This doesn't mean that the concept of repeated addition is wrong. If you go back through the diagrams, you can see that it's sitting there in plain sight. But it simply does not have the flexibility that these other images do. Even though we did not explore them in this section, the concepts of area and A groups of B all extend into fractions and decimals.

In fact, we already saw A groups of B when we were working with fraction multiplication. The product $\frac{2}{3} \cdot \frac{5}{4}$ was two-thirds of a collection of 5 wedges of size $\frac{1}{4}$. And we were able to obtain that value by taking the $\frac{3}{4}$ wedges, cutting them into 3 pieces, and taking 2 out of each of those new collections.

It turns out that area also works with fractions, though you do need to be a bit more careful with your drawings to know where your integers are. Here is the representation of the product $\frac{2}{3} \cdot \frac{5}{4}$ as an area.



Notice that inside of each unit square there are 12 pieces, and that a total of 10 pieces have been shaded in. This means that the final result is $\frac{10}{12}$ (which reduces to $\frac{5}{6}$). If you were to multiply straight across, you would get the exact same result.

These are connections that are both obvious and non-obvious at the same time. If you understand the idea of multiplication being represented by areas, and if you understand parts of a whole, this picture makes complete sense. But if you struggle with one concept or the other, this can seem extremely mysterious and unnecessarily complicated. But that's the tension of mathematical thinking. As you grow in your intellectual sophistication, you can start to see connections arise in a very natural way. But if you don't have the complete toolbox, it's easy to simply be overwhelmed and frustrated. As we approach the last part of this book, we hope that you are becoming more and more like the former than the latter.

30.8 Going Deeper: Alternative Multiplication Techniques

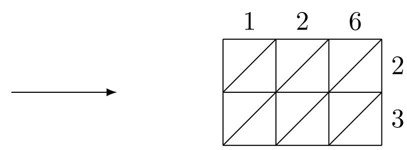
On Worksheet 5, we looked at how the standard multiplication algorithm (multiplication in columns) has all of the same components as looking at multiplication using the grid method we used for algebra. This grid method also has a connection to the base-10 diagrams that were used on Worksheet 4. We're going to look at a couple other multiplication methods. The first is another representations of the same idea, and the second one is a surprising method discovered by the ancient Egyptians that uses only multiplication by 2 with addition.

The Lattice Method:

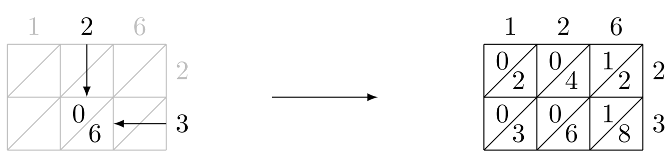
The lattice method is an organizational scheme for calculating products. This method is convenient because it tracks the place values for you, so that you don't need to keep track of the trailing zeros in your numbers. Interestingly, this method of calculation was independently discovered by Arab, European, and Chinese mathematicians.

To use the lattice method, you start by creating a grid where the number of columns is the number of digits in the first number of the product, and the number of rows is the number of digits in the second column. Then write the digits of the numbers in the correspond positions. Lastly, draw the up-right diagonals through all of the boxes. Here is an example:

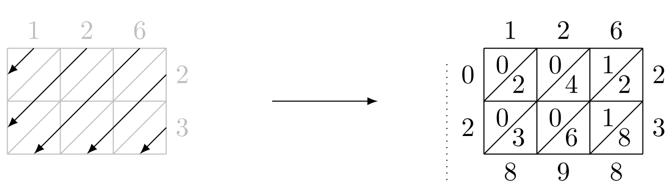
126×23



From here, you write out the products of each pair of numbers in the corresponding square, where the tens digit goes in the upper-left region and the ones digit goes on the lower-right. If your product gives a one-digit number, use a 0 in the tens digit.

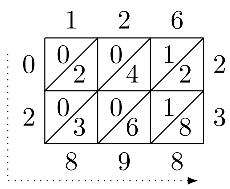


Next, you add along the diagonals down and to the left. Start from the lower-right and work your way to the upper-right. If you end up with a number larger than 10, carry the tens digit to the next diagonal. The final answer is the collection of digits read from the top-left down, then to the right.



$126 \times 23 = 2898$

Although the presentation of this calculation feels significantly different, it's actually the exact same concept that we used with the grid. To see the connection, we simply need to rearrange how we calculate the products. The primary distinction between these two methods is that the place values are being tracked by the diagonals in the lattice method, whereas the grid method requires you to keep track of that yourself. This makes the final addition step significantly more compact.



100	20	6	
2000	400	120	20
300	60	18	3

$$\begin{array}{r} 2000 \\ 300 \\ 400 \\ 60 \\ 120 \\ +\quad 18 \\ \hline 2898 \end{array}$$

Ultimately, neither method is necessarily better than the other. They both accomplish the same thing, and there are costs and benefits to each. The lattice method is similar to the traditional multiplication algorithm in that there are some ideas that get lost in the process. If you were just following the "rules" of the calculation, you may not necessarily see the connection between the diagonals and the place values. On the other hand, it's hard for the grid method to compete with the organization provided by the lattice method.

The Egyptian Multiplication Method:

The Egyptian multiplication method is named this way because they were the first known culture to have adopted this method. There is a similar method called the Russian peasant multiplication method, which was a rediscovery of the method by Russian peasants in the 19th century. This method is significantly different from our other methods because the only multiplication is multiplication by 2. The simplicity of the calculations is part of the appeal of the method.

It's interesting just how often mathematical ideas are discovered (or created) by multiple people at different times and different places.

We will perform the same calculation as before: 126×23. We start by making two columns. The first column starts with the number 1 and the second column starts with the larger number in our product.

1	126

From here, we're going to double each entry moving down the column, stopping when the first column will exceed the second number in the product.

1	126
2	252
4	504
8	1008
16	2016

If we double 16 we get 32 and that is larger than 23.

Next, we look for a combination of numbers in the first column that add up to the second number in the product. The method is quite simple, but explaining it can be a little wordy. The basic idea is that you work your way back up the table, keeping a running total as you go. If keeping the value puts you above your target, skip it. Otherwise, you keep it. At some point, you will hit your target number, and you can skip any remaining values. The process for this example is performed in steps below.

$$\begin{array}{cc|c} & 1 & 126 \\ & 2 & 252 \\ & 4 & 504 \\ & 8 & 1008 \\ \rightarrow & 16 & 2016 \\ \cline{2-3} & 23 & \end{array} \qquad \begin{array}{cc|c} & 1 & 126 \\ & 2 & 252 \\ & 4 & 504 \\ \times & 8 & 1008 \\ \rightarrow & 16 & 2016 \\ \cline{2-3} & 23 & \end{array} \qquad \begin{array}{cc|c} & 1 & 126 \\ & 2 & 252 \\ \rightarrow & 4 & 504 \\ \times & 8 & 1008 \\ \rightarrow & 16 & 2016 \\ \cline{2-3} & 23 & \end{array}$$

$$16 < 23 \qquad 16 + 8 \not< 23 \qquad 16 + 4 < 23$$

$$\begin{array}{cc|c} & 1 & 126 \\ \rightarrow & 2 & 252 \\ \rightarrow & 4 & 504 \\ \times & 8 & 1008 \\ \rightarrow & 16 & 2016 \\ \cline{2-3} & 23 & \end{array} \qquad \begin{array}{cc|c} \rightarrow & 1 & 126 \\ \rightarrow & 2 & 252 \\ \rightarrow & 4 & 504 \\ \times & 8 & 1008 \\ \rightarrow & 16 & 2016 \\ \cline{2-3} & 23 & \end{array}$$

$$16 + 4 + 2 < 23 \qquad 16 + 4 + 2 + 1 = 23$$

The last step is to add up the values on the right side of the chart corresponding to the selected rows. For clarity, we're going to rewrite the desired values before adding.

$$\begin{array}{cc|ccc} \rightarrow & 1 & 126 & \rightarrow & 126 \\ \rightarrow & 2 & 252 & \rightarrow & 252 \\ \rightarrow & 4 & 504 & \rightarrow & 504 \\ \times & 8 & 1008 & & \\ \rightarrow & 16 & 2016 & \rightarrow & 2016 \\ \cline{2-3} \cline{5-5} & 23 & & & 2898 \end{array}$$

For most people, it's quite surprising that this works. Let's take a look at what's actually happening. In our example, notice that the right side is always 126 times the left side. For example, the row with 8 on the left side has $8 \times 126 = 1008$ on the right. We then picked the rows on the left that add up to 23, giving us $23 = 16 + 4 + 2 + 1$. We can take this and multiply both sides by 126:

$$\begin{aligned} 23 \times 126 &= (16 + 4 + 2 + 1) \times 126 \\ &= (16 \times 126) + (4 \times 126) + (2 \times 126) + (1 \times 126) \\ &= 2016 + 504 + 252 + 126 \end{aligned}$$

We can see that the third line consists of the same values that appeared in the right-most column.

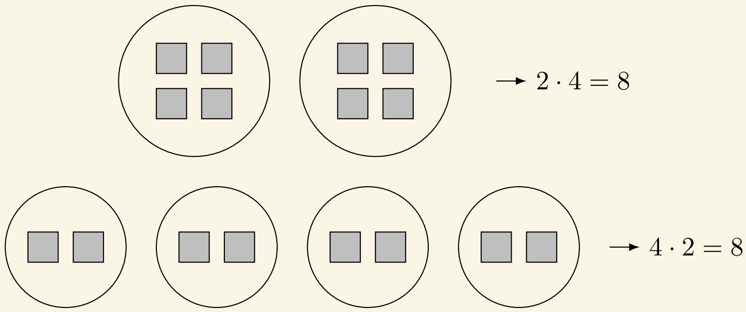
From a mathematical perspective, there's not much difference between the groupings that are being used in this method and the base-10 groupings. All we've done is broken up the product into smaller pieces. In fact, we could even go as far as saying that the only difference between this method and the multiplications we've done previously is that this method is using a binary (or base-2) approach. So even though it looks very different, the underlying concepts are actually still the same.

The point of showing you these examples is to further expand your perspective of multiplication. What is normally taught as a very rigid process turns out to have many different approaches.

Another method is known as Vedic multiplication, which is similar to the standard algorithm but with a fancier organizational scheme.

30.9 Solutions to the “Try It” Examples

1



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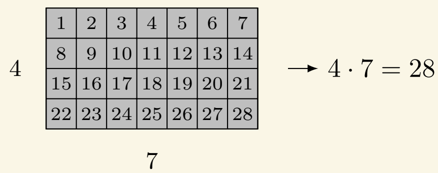

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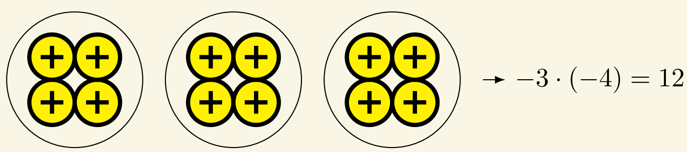
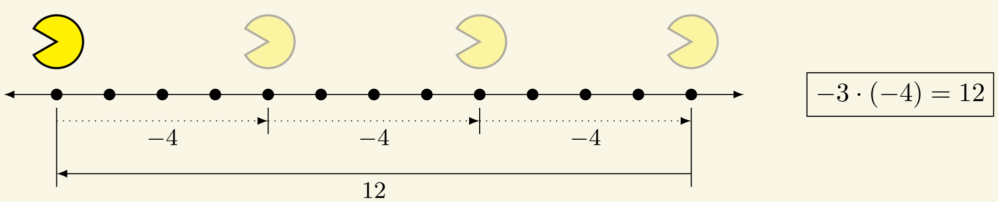


3 groups of the “opposite” of -4



3 groups of the “opposite” of a movement of -4

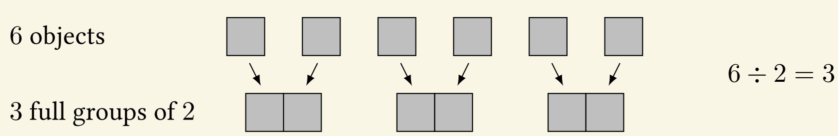
Breaking Down the Mantra: Division

Learning Objectives:

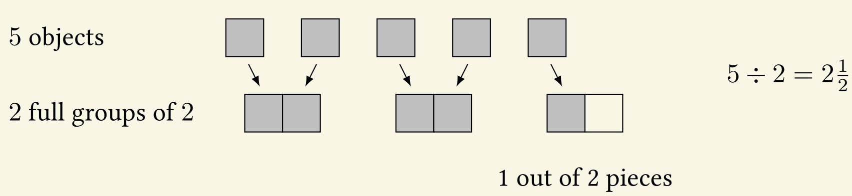
- ☐ Understand division as making groupings.
- ☐ Understand division as equal distribution.
- ☐ Understand why division by 0 is prohibited.
- ☐ Understand the idea behind the long division algorithm.
- ☐ Perform mental division calculations correctly.

So far, we've covered addition, subtraction, and multiplication, which leaves us with division. Division, as with all of the other arithmetic operations, has multiple interpretations and visualizations. The two interpretations come out of the fact that A groups of B and B groups of A both have the same numbers of objects in them.

1 We will start by looking at division as making groupings. This was the way we looked at division earlier when we were working with fractions. The calculation $a \div b$ (which is the same as $\frac{a}{b}$) gives the number of groups of size b that can be made if you start with a objects. Here is the diagram we used to represents $6 \div 2$ from earlier:



This way of looking at division leads to a very natural interpretation of fractions as parts of a whole. If we are short pieces to make a full group, then we use that to determine the fractional part.

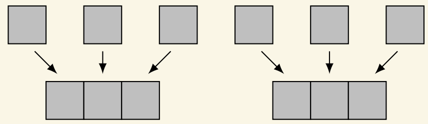


Try it: Draw a diagram that shows the calculation $12 \div 3$ using the concept of making groups.

2 An alternative perspective for division known as equal distribution. The idea here is that you are attempting to create a specific number of equal-sized groups. In this case, $a \div b$ means to determine how many elements will be in each group if you make b equal groups. Here is a diagram:

6 objects

3 in each of 2 groups

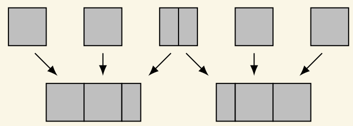


$6 \div 2 = 3$

Sometimes, in order to create equal groups, you need to break some of pieces into parts. In this case, the fractional part comes not from having an incomplete group, but the necessity of dismantling a whole object in order to allow everyone to have an equal share.

5 objects

$2\frac{1}{2}$ in each of 2 groups



$5 \div 2 = 2\frac{1}{2}$

Try it: Draw a diagram that shows the calculation $12 \div 3$ using the concept of equal distribution.

Both of these concepts of division can be used to understand why division by 0 is undefined. Let's look at the meaning of the division calculation $5 \div 0$ as an example. Using the idea of groupings, the question is "How many groups of size 0 are needed to use up 5 objects?" And with groupings of size 0, you're never going to use up all of the objects. If we use equal distribution, the question is "How many objects does each person get if there are 0 people?" Again, the question really doesn't make sense. How many objects does nobody have?

And so this "rule" that dividing by 0 is "undefined" is just representation of the idea that the division concept doesn't make sense when you divide by 0. It is "undefined" because there is no meaningful answer to the question.

Both of these perspectives of division are valid and they are related to each other. We are going to be focusing on making groupings because the connection to parts of a whole is stronger.

We will explore the relationship between them in the worksheets, but it's worth taking a moment to think about it yourself.

Many students have memorized the long-division mantra at some point: "Divide, multiply, subtract, bring down, repeat." And by learning to execute these steps, students learn how to perform long division problems. But what is actually happening as these steps are being executed? Very few people are able to explain what is happening conceptually. And since the emphasis of this book is mathematical thinking and understanding, we're going to break it down and take away the mystery of long division.

The problem of long division is that you're given a large collection of objects and asked how many groups of a specific size can be made from that collection. One way to get the answer is by "counting up" to that value. For example, if we wanted to know how many groups of 4 could be made from 24 objects, we could simply count up to it by doing multiples of 4:

Multiples of 4:	4	8	12	16	20	24
Count:	1	2	3	4	5	6

And from this, we can see that we can make 6 groups of 4 out of 24 objects.

This theoretically works for any number, but it's quickly seen as grossly inefficient. Let's say that we were given 336 objects instead. Trying to count as above is not a smart approach because it will simply take a very long time to get the target number.

Multiples of 4:	4	8	12	16	20	24	28	32	36	40	···
Count:	1	2	3	4	5	6	7	8	9	10	···

Here is where we can invoke our mathematical thinking and problem solving skills. We want to count faster, but we want to do it in an organized manner. Based on our knowledge of numbers, a reasonable approach would be to count in groups of 10 instead of groups of 1. We can quickly see that a group of 10 requires 40 pieces, and use those groupings instead to speed up the process.

Multiples of 4:	40	80	120	160	200	240	280	320	~~360~~
Count:	10	20	30	40	50	60	70	80	~~90~~

Notice that we crossed out the last one because 360 is larger than 336, so we don't have enough pieces to make another 10 groups. But we're also not done yet because we've only accounted for 320 out of the 336 pieces. We can see that there are 16 pieces left, which we know corresponds to another 4 groups. And so in total, there are $80 + 4 = 84$ groups of 4 that can be made with 336 objects.

3 One of the challenges with these calculations is figuring out the "correct" amount of work to show. With practice, these calculations can be done mentally without too much difficulty. But it takes some time to get there. If you wanted to write out multiples of numbers as above as scratch work, that would be acceptable. But for the purposes of these division problems, the following is the minimum presentation expectation:

$$\left.\begin{aligned} 80 \cdot 4 &= 320 \\ 4 \cdot 4 &= 16 \end{aligned}\right\} \implies 336 \div 4 = 84$$

This diagram represents the creation of the different sized groupings as well as shows how they come together to give the final answer.

Try it: Determine the value of $462 \div 6$ using the method described above.

The long division method is precisely this grouping process, but written in a far more compact style. The trick to unwinding it is to think past the digits and contemplate the process as working with numbers. We will follow the steps of the long division calculation $336 \div 4$ and track the logic of the calculation we just completed.

$$4\,\overline{)\,\underline{3}\,3\,6}$$

Long division: Does 4 go into 3? No.
Conceptual: How many groups of 400 can be made from 300? None.

$$\begin{array}{r} 8 \\ 4\,\overline{)\,\underline{3\,3}\,6} \end{array}$$

Long division: Does 4 go into 33? Yes. 8 times.
Conceptual: How many groups of 40 can be made from 330? 8 groups.

$$\begin{array}{r} 8 \\ 4\,\overline{)\,3\,3\,6} \\ -\,3\,2\downarrow \\ \hline 1\,6 \end{array}$$

Long division: Multiply, subtract, bring down.
Conceptual: After making those groups, how many are left? 16 pieces.

$$\begin{array}{r} 8\,4 \\ 4\,\overline{)\,3\,3\,6} \\ -\,3\,2 \\ \hline 1\,6 \end{array}$$

Long division: Does 4 go into 16? Yes. 4 times.
Conceptual: How many groups of 4 can be made from 16? 4 groups.

So the underlying logic of long division is to make the big groups first and then work your way down to smaller groups.

4 Something else happens when you start to open up that logic. The mental calculations actually become much simpler. In the same way that it is difficult for your brain to keep track of all of the symbols when adding in columns, long division is an extremely complicated process if you need to remember both the values and locations of all the digits. But if you think about them as numbers, you significantly simplify the logic and (with a little bit of practice) this becomes a much simpler mental calculation.

Try it: Mentally calculate $462 \div 6$.

31.1 Division - Worksheet 1

1 Draw a diagram that shows the calculation $24 \div 6$ using the concept of making groups.

2 Draw a diagram that shows the calculation $24 \div 6$ using the concept of equal distribution.

3 Draw a diagram that shows the calculation $21 \div 3$ using the concept of making groups.

4 Draw a diagram that shows the calculation $21 \div 3$ using the concept of equal distribution.

31.2 Division - Worksheet 2

1 Draw a diagram that shows the calculation $15 \div 3$ using the concept of making groups.

2 Draw a diagram that shows the calculation $15 \div 3$ using the concept of equal distribution.

3 Draw a diagrams that represent the product $3 \cdot 5$ as 3 groups of 5 and as 3 groups of 5.

4 Based on the diagrams that you've created, pair up each division concept with one of the A groups of B diagrams. This shows how both division concepts represent the same multiplication calculation.

31.3 Division - Worksheet 3

1 Determine the value of $248 \div 4$, showing your work as described in this section.

2 Determine the value of $252 \div 7$, showing your work as described in this section.

3 Determine the value of $1432 \div 4$, showing your work as described in this section.

Notice that you can go bigger than groups of 40.

4 Practice your mental arithmetic by performing the following calculations.

$114 \div 3 =$	$385 \div 5 =$	$192 \div 8 =$
$294 \div 6 =$	$266 \div 7 =$	$423 \div 9 =$
$738 \div 6 =$	$2358 \div 3 =$	$2198 \div 7 =$

31.4 Division - Worksheet 4

Follow the logic and do not be afraid that you are dividing by a two-digit number.

1. Determine the value of $532 \div 14$, showing your work as described in this section.

2. Determine the value of $598 \div 13$, showing your work as described in this section.

3. Determine the value of $756 \div 21$, showing your work as described in this section.

4. Determine the value of $990 \div 18$, showing your work as described in this section.

31.5 Division - Worksheet 5

1

Sometimes, a division calculation is "close" to being something extremely easy. For example, the calculation $495 \div 5$ is very close an easily calculated $500 \div 5$. But since it's easy to see that $500 \div 5$ is 100 and that 495 is one group of 5 less than 500, it is not too difficult to see that $495 \div 5 = 99$ (1 group short of 100 groups of 5).

This method of finding approximate answers and adjusting is commonly done in mental arithmetic when it's possible. But it's a good way to quickly calculate the answer in those situations.

Practice your mental arithmetic by performing the following calculations.

$297 \div 3 =$	$995 \div 5 =$	$792 \div 8 =$
$1194 \div 6 =$	$2691 \div 9 =$	$693 \div 7 =$
$1592 \div 8 =$	$2093 \div 7 =$	$1996 \div 4 =$

2

It's not always the case that the number will be "one away" from a nice value. Sometimes, it may be two or three away. But with some practice and experience, you can learn to spot those values, too.

Practice your mental arithmetic by performing the following calculations.

$1490 \div 5 =$	$1782 \div 6 =$	$392 \div 4 =$
$882 \div 9 =$	$3486 \div 7 =$	$1188 \div 6 =$
$1491 \div 3 =$	$1576 \div 8 =$	$5982 \div 6 =$

3

Sometimes the "nice" value isn't quite as nice, but is still helpful. Instead of being near a multiples of 100, these are examples that are near multiples of 10. The same ideas still apply. These are a bit trickier because you really need to be comfortable with your multiplication table to do it.

This isn't a race! Focus on thinking through the process carefully and correctly.

$195 \div 5 =$	$236 \div 4 =$	$343 \div 7 =$
$354 \div 6 =$	$312 \div 8 =$	$342 \div 9 =$
$476 \div 7 =$	$531 \div 9 =$	$456 \div 8 =$

31.6 Deliberate Practice: Division

Focus on these skills:

- Think through the division by making groupings.
- Perform the calculation mentally before writing out the minimal presentation in this section.
- Present your work legibly.

Instructions: Perform the indicated calculation.

1 $372 \div 6$

2 $296 \div 8$

3 $265 \div 5$

4 $273 \div 7$

5 $342 \div 9$

6 $512 \div 8$

7 $402 \div 6$

8 $504 \div 9$

9 $315 \div 5$

10 $294 \div 7$

31.7 Closing Ideas

We have now completed a tour of all four arithmetic operations: addition, subtraction, multiplication, and division. We have seen that addition and subtraction are closely related to each other, and that multiplication and division are closely related to each other. This touches on an idea that was discussed much earlier, but is worth reviewing now that we have more knowledge and experience.

In the early parts of this book, we introduced certain properties of addition and multiplication:

- The commutative properties: $a + b = b + a$ and $a \cdot b = b \cdot a$
- The associative properties: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

The fact that these properties hold for addition and multiplication, but not for subtraction and division, is a signal that addition and multiplication are somehow more "basic" or more "fundamental" than the other operations. And if we think back to the work that we've done over the last several sections, we can start to see how that has played out in our analysis.

Addition is closely tied to counting. When we add numbers, it can be viewed as a process of counting up by a certain amount. We saw this when we were combining groups of objects together (whether using base-10 blocks, integer chips, and even movements on the number line). But when we subtracted, we were still required to count up to a number in some form. We had to count out the right number of negative chips, or we had to count up to the right number of objects to take away, or we had to count up to the right number of movements. And so in all situations, our core addition concept was part of the process of subtracting.

Something similar happened with multiplication and division. The process of division required us to make groups that built up to a specific value. We saw this explicitly when we were counting out multiples of numbers in the division problems. We also saw this visually when we took individual objects and turned them into groups of objects (multiplication is represented as groups of objects).

As you continue onward in your college level mathematics, you may start to see further hints of this idea. For example, in the study of logarithms, you'll see that the primary relationship is between addition and multiplication, and the relationship between subtraction and division turns out to be nothing more than a fancy way of rewriting that relationship. Those calculations are included below. It's okay if you do not really understand it right now. Just focus on the relationships between addition and multiplication and how they relate to the relationships between subtraction and division.

- Sum of Logarithms: $\log(a) + \log(b) = \log(a \cdot b)$.
- Difference of Logarithms: $\log(a) - \log(b) = \log\left(\frac{a}{b}\right)$.

31.8 Going Deeper: Remainders and Decimals

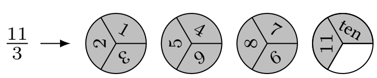
In this section, we mostly avoided remainders so that the focus would be on the primary concept of division. We're going to take a deeper dive into this topic to see how this idea relates back to one of the core concepts of fractions, and also take a deeper look at the decimals associated with those fractions.

It's often the case that division problems don't work out evenly. In elementary school, children learn that when you have leftover pieces, it's called a remainder. And in the long division process, they are often taught to write their answers with an R, so that a calculation like $11 \div 3$ results in $3\,\mathrm{R}2$. What's interesting about this notation is that there's no other place in mathematics where we use it.

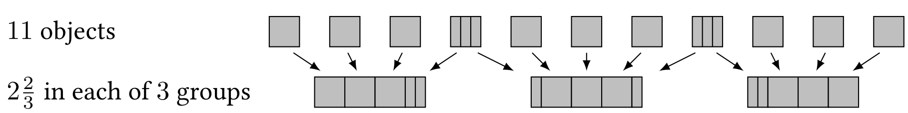
Remainders are useful in certain areas of mathematics, so it's not that the idea isn't used. But we don't use the R notation in those other settings.

The main tool for working with remainders is the idea of parts of a whole, which we discussed in detail in Section 17. The basic idea is that the fraction $\frac{a}{b}$ means to take a whole unit (often represented by a circle) and cut it into b pieces, and then the quantity $\frac{a}{b}$ is the amount you would have if you had a of those pieces. breaking a whole unit into pieces allows us to make groupings and equal-sized distributions when the calculation doesn't work out evenly.

For example, in the calculation $11 \div 3$, if we were thinking about making groupings, we would have three full groups and two out of three pieces to make another group. The language of "two out of three pieces" is exactly mirroring the language we use with parts of a whole. Here's what $\frac{11}{3}$ looks like as a parts of a whole diagram:



This gives us the explicit image of having three full groups and two out of three pieces needed for another group. If we wanted to view this as equal distribution, we would have a diagram that looks more like this:



In both situations, the need for having parts of a whole comes up naturally.

In Section 17, there were a couple exercises on the worksheet that discussed the conversion of fractions to decimals. In those sections, we simply assumed that you had sufficient familiarity with decimals that you would be able to come up with the answers and they would make sense to you. But we didn't really talk about what was going on with those decimals.

Recall that decimals are really just fractions with powers of ten in the denominator. And so in some sense, decimal expansions are just another way of trying to express a division calculation. In most practical settings, it's far more likely that remainders will be expressed as decimals instead of fractions. There are many reasons for this, including the simple fact that we use computers and calculators and they generally give us decimal answers.

Some fractions have *terminating* decimal expansions. This means that at some point the decimal stops and gives us an exact answer. The simplest examples are the ones where we start with a power of ten in the denominator. For example, $\frac{3}{10} = 0.3$ and $\frac{51}{100} = 0.51$. These are quite literally the direct translation of fractions to decimals.

In other cases, we may not initially have a power of ten in the denominator, but we can get there by finding a different representation of that fraction. Here are a couple examples of this:

$$\frac{1}{2} = \frac{1 \cdot 5}{2 \cdot 5} = \frac{5}{10} = 0.5 \qquad \frac{3}{4} = \frac{3 \cdot 25}{4 \cdot 25} = \frac{75}{100} = 0.75$$

In both of these situations, there's a way to cut up our parts of a whole diagram so that we have a power of ten pieces. We can visualize this with $\frac{1}{2}$, but with $\frac{3}{4}$ it's very difficult to see all 100 subdivisions so we didn't draw all of them. But you should get the general idea from the diagram.

$\frac{1}{2}$ $\quad$ $\frac{5}{10}$ $\quad$ $\frac{3}{4}$ $\quad$ $\frac{75}{100}$

Next, we have fractions for which we simply cannot use integers to get that representation. For example, with the fraction $\frac{1}{3}$, there is no integer value x so that $3x$ is equal to a power of ten:

$$\begin{aligned}
3x &= 1 & &\implies x = \tfrac{1}{3} \\
3x &= 10 & &\implies x = 3\tfrac{1}{3} \\
3x &= 100 & &\implies x = 33\tfrac{1}{3} \\
3x &= 1000 & &\implies x = 333\tfrac{1}{3} \\
3x &= 10000 & &\implies x = 3333\tfrac{1}{3} \\
3x &= 100,000 & &\implies x = 33333\tfrac{1}{3} \\
3x &= 1,000,000 & &\implies x = 333,333\tfrac{1}{3}
\end{aligned}$$

We can see that no matter how far out we go, we'll never get an integer answer.

These values were written as mixed numbers for a reason. That reason is that it makes it clear that even though there is no integer value, there's a definite pattern that's developing. There is some number of 3s, but then there's always one out of three pieces left over. And this happens over and over again. Because of this, we know that the decimal expansion for $\frac{1}{3}$ is a *repeating decimal*.

The idea of a repeating decimal is that there is some pattern of numbers that repeats indefinitely. The pattern is not limited to a single digit, nor does it have to start immediately after the decimal. The feature is that the decimal eventually falls into a fixed pattern. There are two notations for repeating decimals. The implicit notation leaves it to the person reading it to identify the pattern. For example, here is the decimal expansion of $\frac{1}{3}$:

The implicit notation can be confusing because we also use ... when we have decimals that don't repeat. For example,

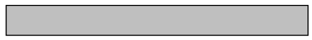
$$\pi = 3.14159265\ldots$$

$$\frac{1}{3} = 0.333333\ldots$$

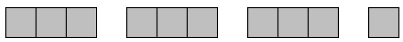
In this case, the pattern is pretty easy to spot. But there's also an explicit notation that specifically marks out the pattern. This is good for patterns that take longer to repeat:

$$\frac{7}{17} = 0.\overline{4117647058823529}$$

We're going to spend some time exploring the nature of these repeating decimals by thinking about fractions. Can we explain why those decimals repeat? If we look at the column of values we had for the fraction $\frac{1}{3}$, we can see that $\frac{1}{3}$ keeps appearing over and over again. The process being shown above in equations is a bit easier to understand in pictures. We are trying to calculate $\frac{1}{3} = 1 \div 3$. We will represent 1 by a bar:



We want to divide this into three equal pieces. However, decimals constrain us to having to break things into ten pieces. So we'll do that and do our best to create three equal groupings without breaking things up any further:



We get three groups and then one leftover piece. In order to try to create equal groupings, we're going to need to break this piece up. But again, the limitation of using decimals is that we can only break this up into ten pieces. But taking one item and breaking it into ten pieces is exactly what we just did, so we already know the result. We're going to end up with three more groupings, and then one leftover piece. And if we were to try to take that leftover piece and break it into ten again, it's just going to be the same thing forever.

This pattern of breaking things into ten, dividing, and looking at the remainder is what leads to repeating decimals. Notice that breaking into ten pieces is the same as multiplying the number of pieces by ten, so that we can interpret this process as multiplying by ten, dividing, and looking at the remainder. We can see that the size of the remainder is what controls the next step in the process.

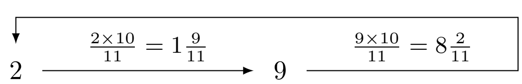
Let's look at another example that has a little but more going on. Let's say we were trying to get a decimal for $\frac{2}{11}$.

$$2 \xrightarrow{\frac{2\times 10}{11} = 1\frac{9}{11}} 9 \qquad\qquad \frac{2}{11} \approx 0.1$$

The 1 of the mixed number is our first decimal value, and the 9 in the numerator is our remainder. We can then repeat this process to get the next decimal:

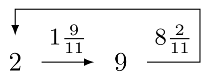
$$2 \xrightarrow{\frac{2\times 10}{11} = 1\frac{9}{11}} 9 \xrightarrow{\frac{9\times 10}{11} = 8\frac{2}{11}} 2 \qquad\qquad \frac{2}{11} \approx 0.18$$

But notice that we have a remainder of 2, which is a number that we've already seen. In fact, we explicitly see the $\frac{2}{11}$ that we started with in the calculation. So we can loop back to that value to create the loop that gives us the repeating decimal expansion!

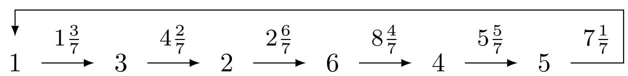


$$\frac{2}{11} = 0.181818\ldots = 0.\overline{18}$$

We're going to condense the notation a bit so that it takes up less space. Since each arrow represents multiplying by 10 and dividing by the denominator, we're just going to give the answers at each step. So the above diagram can be reduced to this one:

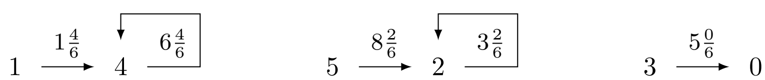


Let's take a look at an example with a longer pattern. Here is what we get when working with $\frac{1}{7}$.



In this case, we have a cycle that runs through every single remainder. This means that we have the full information about the decimal expansions when the denominator is 7. If we wanted the decimal expansion of $\frac{2}{7}$, we start from the 2 and write down the integer parts of the mixed numbers in order: $\frac{2}{7} = 0.\overline{285714}$.

Different denominators will have different diagrams. Here is what the denominator is 6 looks like. Notice that we don't reduce the fractions (because that changes the value of the numerator and the denominator).



$$\frac{1}{6} = 0.1\overline{6} \qquad \frac{2}{6} = 0.\overline{3} \qquad \frac{3}{6} = 0.5 \qquad \frac{4}{6} = 0.\overline{6} \qquad \frac{5}{6} = 0.8\overline{3}$$

In this case, there are three separate pieces, two of which repeat (but don't repeat the entire chain) and one that terminates. So it's possible to get denominators that multiple possibilities.

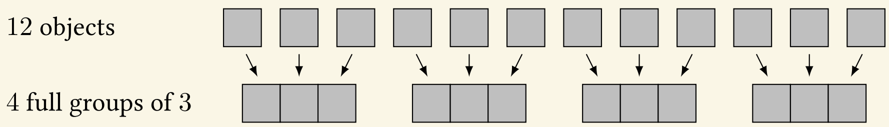
Here are a few questions to consider. If you were to be given some other denominator (such as 12),

- Can you predict how many fractions will terminate and how many won't?
- Can you predict how many different cycles you will have?
- Can you predict which cycles will have full loops and which ones will have partial loops?

You actually have all the tools you need to explore these questions. You may even be able to come up with some conjectures and have some explanations for why you think your ideas are correct. At this point, you may not have the tools to fully explain everything, but that's part of the learning process. As you get interested in new ideas, that's the starting point for developing new tools and new skills to try to find out more.

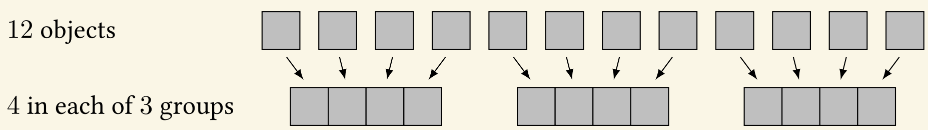
31.9 Solutions to the "Try It" Examples

1



$$12 \div 3 = 4$$

2



$$12 \div 3 = 4$$

3

$$\left.\begin{aligned} 70 \cdot 6 &= 420 \\ 7 \cdot 6 &= 42 \end{aligned}\right\} \implies 462 \div 6 = 77$$

4

$$462 \div 6 = 77$$

Yet Another Pause for Reflection

In the third branch, took a long look at some basic arithmetic concepts that you have probably known for the vast majority of your life. But we looked at these things in ways that you may have never seen before in your life. This is an important process in our intellectual growth as humans. There is value to periodically returning to some basic ideas to explore them again with the added knowledge and insights that can be gained only through years of experience. The discovery of new insights from old ideas drives a wide swath of mathematical thinking.

The big emphasis of this branch was to expand your mental toolbox for working with numbers. Students that can only work with numbers as manipulations of digits tend to be less prepared to make conceptual connections than those that have multiple ways of thinking about numbers. In particular, being able to relate arithmetic to geometric concepts is useful in practical applications. For example, when working with information on a coordinate grid, knowing that the distance between two points is the difference between them brings insight into certain mathematical formulas, such as the point-slope form of a line, the distance formula, and the equations of circles.

In this portion of the course, we have covered the following topics:

- The Number Line and Base-10 blocks as Visualizations of the Integers
- Visualizations of Addition and the Addition Algorithm
- Visualizations of Subtraction and the Subtraction Algorithm
- Integer Chips as a Representation of Negative Numbers in Addition and Subtraction Calculations
- Movement on the Number Line for Negative Numbers in Addition and Subtraction Calculations
- Visualizations of Multiplication as A Groups of B and Area
- Visualizations of Division as Making Groupings and Equal Distribution

Questions About the Content

1 Were there any topics that you had seen before, but you understand better as a result of working through it again?

2 Were there any ideas that you had never seen before?

3 Based on your experience, which of these ideas seems the most important to understand well?

4 Did any part of the presentation make you curious about math in a way that went beyond the material? Are there questions or ideas that you would like to explore?

Examples:

- Are there other visualizations of numbers?
- Are there other arithmetic algorithms?
- Are there any other "tricks" to mental arithmetic?

Questions About You

1 How has your mathematical thinking continued to evolve from the previous pause for reflection? Do you find yourself thinking in different ways?

2 Did you have any "Aha!" moments where you had an insight into something that you had not noticed before?

3 What is the biggest mathematical connection that you made?

4 How is your mathematical confidence coming out of these sections? Is it going higher, lower, or staying at about the same as it was?

Key Words are (Mostly) Evil: Mathematical Relationships

Learning Objectives:

- ☐ Translate verbal relationships into mathematical relationships.
- ☐ Find mathematical relationships that match given data.

Students tend to dislike word problems. This is probably because word problems ask students to do something more difficult than pure calculations. Word problems ask students to understand information and then translate it into a form that allows them to use their mathematical toolbox.

In order to help students with this transition, some teachers started to introduce rule-based patterns into reading word problems. The problem is that this has led to students thinking incorrectly about the process of thinking through word problems. This approach is sometimes called "key words" and you can find lists of "key words" that are categorized by operation.

If you've been following along with the challenges of rule-based thinking that we've been discussing throughout this book, you'll know where we're going.

The problem with this is that these lists are (mostly) evil. There is a grain of truth in them, but it sells students short of an actual understanding of how to think through word problems. It causes students to read word problems in ways that are sometimes incorrect and does not advance the cause of mathematical thinking.

The grain of truth is that the words do indicate mathematical relationships that involve the operation. But that's different from treating the word as a rule.

1

Here is a classic example: "Bob has 5 rocks. Alice has 3 more rocks than Bob. How many rocks does Alice have?" The common thought process for people is that "more" tells us we need to add, so Alice has 8 rocks. The answer is correct, but the reasoning is wrong. But to understand why the reasoning is wrong, we need to look at a slightly different problem.

Try it: Alice has 5 rocks. Alice has 3 more rocks than Bob. How many rocks does Bob have? Explain your reasoning.

In the example, the mathematical operation that was required was subtraction, but "more" means to add (if you believe in the rule-based key word approach). And so this leads us to the question of what's really happening in the word problem.

2

In most word problems, the words do not directly describe mathematical operations that you need to perform. They are not instructions. Rather, they are representations of mathematical equations using words. In the example above, the relationship is "Alice has 3 more rocks than Bob." We can translate this into the following:

$$a = \text{The number of rocks Alice has}$$
$$b = \text{The number of rocks Bob has}$$

$$a = b + 3$$

There are some key points to remember. The first is that the variables used are all defined. This is an important element of communication. If you introduce a symbol, you need to be clear what the symbol means. Furthermore, it's not acceptable to say that a represents "Alice."

Variables are symbols that represent a quantity or a mathematical expression. Alice is a person, not a number. So it is good to get into the habit of being accurate in describing your variables.

Once we have the relationship established, we can then move to plug in the variables. If we are told that "Bob has 5 rocks" then we can set $b = 5$ and solve:

$$\begin{aligned} a &= b + 3 \\ &= 5 + 3 && \text{Substitute } b = 5 \\ &= 8 && \text{Arithmetic} \end{aligned}$$

Or we are told that "Alice has 5 rocks" then we can set $a = 5$ and solve:

$$\begin{aligned} a &= b + 3 \\ 5 &= b + 3 && \text{Substitute } a = 5 \\ b &= 2 && \text{Subtract 3 from both sides} \end{aligned}$$

The actual decision to add or subtract is based on the relationship that is established by the words, not through identifying a specific word as if it were an instruction. This distinction once again highlights mathematical thinking. You're not looking for a specific rule or algorithm to follow, you're looking to understand a relationship so that you can apply the appropriate tool to solve your problem.

If you are unsure about the mathematical relationship you've created, it can always be checked by plugging in a couple numbers. There is clarity in examining specific values that can sometimes be lost when looking at symbols. For example, when reading $a = b + 3$ some students will think that b is the bigger value "because 3 is being added to it." But this is not true, and the clearest way to see it is to just pick a value of b and see what the formula actually gives as the result.

Try it: Annika is 4 inches taller than Carlos. Write an equation that describes this relationship.

Implicit in writing the relationship is that you should define your variables.

3 Once a mathematical relationship is established, it then becomes the foundation for solving a word problem based on that relationship.

Try it: Annika is 70 inches tall. She is also 4 inches taller than Carlos. How tall is Carlos?

Students shy away from word problems because of the extra step of effort that's required to translate the words into mathematical symbols. However, that extra step is the step that covers the gap between "real life" and mathematics. Making that connection is an important piece of the puzzle for mathematical thinking.

In addition to having mathematical relationships described by words, there are situations where the relationship is described by a chart of values.

x	y
1	2
2	4
3	6

After some thought, the pattern that you will likely see is that $y = 2x$. And you can check this by plugging in values.

It turns out that there are all sorts of formulas that would match those values. If you were to use the equation $y = -\frac{1}{6}x^3 + x^2 + \frac{1}{6}x + 1$, you would find that it also works! However, for these problems we are going to stick with the "obvious" answer and not worry about more complicated relationships.

You might want to try to figure out how we got that other equation.

4 When trying to find a formula that matches a given table, there is no magic formula. And while there is a computational method if you can guess the "form" of the equation, it's better to develop intuition. The relationships that you will be given to work with are going to be relatively simple, and you should be able to come up with them by trying to think logically and in an organized manner.

Try it: Find a formula that relates x and y based on the following chart.

There are multiple ways to represent the relationship. Just make sure that your form works for the given values.

x	y
1	9
4	6
8	2

33.1 Mathematical Relationships - Worksheet 1

1 Matt and Reuben have 14 toys between the two of them. Write an equation that describes this relationship.

Remember to define your variables!

2 Matt and Reuben have 14 toys between the two of them. Matt has 5 toys. How many toys does Reuben have?

3 Reuben has 4 more toys than Matt. Write an equation that describes this relationship.

4 Reuben has 4 more toys than Matt. Together, they have 14 toys between them. How many toys does each one have?

Use the relationship as a substitution.

33.2 Mathematical Relationships - Worksheet 2

1 A box of cans of soda contains 12 cans. Write a relationship between the number of boxes and the number of cans of soda.

2 A store has a sale on boxes of cans of soda. However, the store limit is 8 boxes per customer. If each box of soda contains 12 cans, how many cans of soda can be bought by a single customer during this sale?

3 Dr. Pepper is a math professor. Every semester, she buys each of her students a can of her favorite soda. She has 30 students in her class. If a box of cans of soda contains 12 cans, how many boxes does she need for her class?

4 In the previous question, there were two possible answers, depending on how you saw the problem. You might have had a fractional box of sodas, or you might have decided a fractional box of soda doesn't make sense and rounded up to an integer number of boxes. Is it possible that both answers are correct, or is one answer correct and the other incorrect?

33.3 Mathematical Relationships - Worksheet 3

1 Find a formula that relates the variables in each chart. (Note: Each chart is separate.)

x	y
1	4
2	5
3	6

x	y
1	3
2	6
3	9

x	y
1	3
2	5
3	7

2 Find a formula that relates the variables in each chart. (Note: Each chart is separate.)

Each of these has relatively simple formulas that only involve integers.

x	y
5	20
15	10
30	-5

x	y
3	7
5	13
8	22

x	y
2	-3
3	-5
6	-11

3

x	y
2	12
4	11
6	10

x	y
5	3
10	4
15	5

x	y
3	1
5	5
9	13

4 Find a formula that relates the variables in each chart. (Note: Each chart is separate.)

The idea for the first one may help you with the others.

x	y
1	1
2	4
3	9

x	y
1	3
2	6
3	11

x	y
1	0
2	-3
3	-8

33.4 Mathematical Relationships - Worksheet 4

1 A gym membership at Gym A costs \$60 as an initiation fee plus \$20 per month. Write a formula that relates the gym costs and the length of the membership.

2 A gym membership at Gym B costs \$100 as an initiation fee plus \$10 per month. Write a formula that relates the gym costs and the length of the membership.

3 Determine the price of 4 months of membership at each gym. Which one is the better deal if you are a member for less than 4 months? Which is the better deal if you are a member for more than 4 months?

4 Suppose you were thinking about starting an exercise routine, and that you were deciding between Gym A and Gym B. You know that you want to give it at least three months of effort, but you're not sure if you'll go beyond that. Which gym would you choose? Explain the logic of your decision.

There is no right answer. You really want to focus on whether your answer is reasonable.

33.5 Mathematical Relationships - Worksheet 5

1 Write a word problem that uses the phrase “more than” but that you cannot just add the numbers in the problem to get the correct answer. Then solve it.

2 Write a word problem that uses the phrase “fewer than” but that you cannot just subtract the numbers in the problem to get the correct answer. Then solve it.

3 Write a word problem that uses multiplication in the mathematical relationship between the variables. Then solve it.

33.6 Deliberate Practice: Basic Word Problems

Focus on these skills:

- Define your variables.
- Write the equation or equations that represent the given mathematical relationship or relationships.
- Present your work legibly.

Instructions: Perform the indicated calculation.

1 Kai and Ama are math tutors. Kai has 12 clients, and Ama has 5 more clients that Kai. How many clients does Ama have?

2 Kai and Ama are math tutors. Kai has 7 clients, which is 3 fewer than Ama. How many clients does Ama have?

3 Kai and Ama are math tutors. Kai has 7 clients, and Ama has four fewer clients than Kai. How many clients do they have in total?

4 Martin and Sylvia have each have marbles in their pockets. Sylvia has 7 more marbles than Martin, and Martin has 9 marbles. How many marbles does Sylvia have?

5 Martin and Sylvia have each have marbles in their pockets. Sylvia has 5 fewer marbles than Martin, and Martin has 8 marbles. How many marbles does Sylvia have?

6 Martin and Sylvia have each have marbles in their pockets. Together, they have 22 marbles. Sylvia has 2 more marbles than Martin. How many marbles do they each have?

7 Leonard and Carlos are having a pickle-eating contest. Carlos ate three fewer pickles than Leonard. Carlos managed to eat five pickles. How many pickles did they eat in total?

8 Leonard and Carlos are having a pickle-eating contest. Carlos ate three more pickles than Leonard. In total, they ate 11 pickles in total. How many pickles did they each eat?

9 A food box program costs $30 per month if you pay as you go, but if you pay for a year's worth of boxes up front, they will give you two months for free. What is your average monthly discount for paying in advance?

10 A food box program costs $30 per month if you pay as you go, but if you pay for a six months at a time, you get two points towards their bonus program. If you get ten bonus points, they will give you a free box. How long will it take to earn a free box?

33.7 Closing Ideas

This is the first section in the final branch of this book. The ideas are about trying to bridge the gap between formal mathematics and applications. In this first section, we focused primarily on relationships between variables because this is the core value of mathematical reasoning. Translating quantitative relationships into mathematical equations is the key skill that employers want when they talk about wanting employees with “math skills.” They don’t need people to just perform calculations, since they have computers and calculators that can do that for them. They need people who can translate the needs of the business into a mathematical language that they can then use to make decisions. The problem of the two gyms in the worksheets is an example. There often isn’t just a single answer. It depends on non-mathematical features of the situation that can’t always be easily quantified.

And that’s an important takeaway for thinking about applications of mathematics. It’s not always the pursuit of “the” answer, but rather it’s a tool to help you think about the situation and provide quantitative information to help with the decision-making process.

33.8 Solutions to the "Try It" Examples

1 Since Alice has more rocks than Bob, Bob has a smaller number of rocks compared to Alice. So if Alice has 5 rocks, we need to subtract 3 to get the number of rocks that Bob has.

Your exact language may vary, but the ideas should be basically the same.

2

$$a = \text{Annika's height in inches}$$
$$c = \text{Carlos' height in inches}$$

$$a = c + 4$$

3

$$a = c + 4$$
$$70 = c + 4 \quad \text{Substitute } a = 70$$
$$c = 66 \quad \text{Subtract 4 from both sides}$$

4

$$x + y = 10$$

Your answer may vary, but it must be equivalent to the one provided.

Numbers That Are Annoyingly Big or Small: Scientific Notation

Learning Objectives:

- ☐ Represent numbers using scientific notation.
- ☐ Evaluate expressions given in scientific notation.

In science courses, there are important values that are sometimes very large or very small. For example, in a course such as astronomy, you might learn that the average distance from the earth to the sun is approximately 149600000000 meters. In chemistry, you will run into a unit called a mole, which represents approximately 602214076000000000000000 molecules. You might also have very small numbers, such as atoms being approximately 0.0000000001 meters in size.

One of the challenges with numbers like these is that it can be very difficult to compare them. For example, determine which of the two numbers is the larger number: 10000000000000000000 or 5000000000000000000. It can be an annoying and frustrating process to have to count out all of the digits in those numbers, and it's easy to make mistakes. Some people will use commas or small spaces every three digits (starting from the decimal point) in order to help, but this is still a lot of writing and opportunities for errors.

This is addressed in multiple ways. We are going to focus on scientific notation in this section. The idea of scientific notation is to take these long numbers and write them in a more compact form that makes them easier to think about and easier to work with.

Another way this is address is by making up new words. No, this is not a joke.

Definition 34.1. *Scientific notation* is a representation of numbers (that are typically very large or very small) using the form $a \times 10^n$, where a is a decimal number between 1 and 10 (including 1 and excluding 10), and n is some integer.

We will be using $\times$ instead of $\cdot$ to avoid confusion as there will be many decimals in this section.

The value of scientific notation is that it replaces the process of counting out zeros and displays that value explicitly. It comes out of the fact that multiplying or dividing a number by a power of 10 corresponds to "moving the decimal point" a certain number of places.

1 The idea of "moving the decimal point" is extremely common when performing these arithmetic calculations. This idea is just a reflection of the fact that we work with a base-10 system for writing down numbers. When multiplying by a positive power of 10, the multiplication makes the final result bigger.

$$4.5 \times 10^3 = 4\underbrace{500}.$$

Original decimal point — 3 places — Final decimal point

Think of this as multiplying by a "big" number.

And when multiplying by a negative power of 10, the multiplication makes the final result smaller.

$$4.5 \times 10^{-3} = 0.0045$$

Final decimal point — 3 places — Original decimal point

Think of this as multiplying by a "small" number.

Try it: Convert the numbers 1.234×10^5 and 9.87×10^{-6} into standard form.

2 Putting a number into scientific notation follows the same idea. However, in this case you must decide the number of positions that the decimal point must move. Some people say that the signs are "switched" when putting the number into scientific notation, but that tends to cause confusion because students often forget which way the correct way is and so when they "switch" they still get it wrong. The best way to avoid errors is simply to take a moment and think about the final answer. If you started with a big number, then your scientific notation should represent a big number. And if you started with a small number, your scientific notation should represent a small number.

The other important factor to remember is that your a value should be a number between 1 and 10, but not 10 itself. Another way to think about it is that you only want one digit to the left of the decimal point (and that digit should not be zero) when you're done.

$$27830000 = 2.783 \times 10^7$$

Final decimal point — 7 places — Original decimal point

Try it: Convert the numbers 3480000000 and 0.0000736 to scientific notation.

Another reason for scientific notation is that it allows us to more easily perform arithmetic. The product $2000000000 \cdot 400000000000$ is definitely an 8 followed by some number of zeros, but counting those zeros runs into the same risks of errors as we saw above.

Try doing this on your calculator. Either your calculator won't be able to handle it or it will automatically give you the answer in scientific notation. So either way, you're going to need to know it.

3 To perform arithmetic with scientific notation correctly, it is helpful to think about some basic algebra ideas. Instead of thinking about $a \times 10^n$, it can be more helpful to think about ax^n, where $x = 10$. The benefit to this is that it allows you to use your past experience and intuition.

We'll start with multiplication. What is $2x^7 \cdot 3x^9$? You might recall that we multiply the number parts and then multiply the variable parts, and that the variable parts satisfy $x^n \cdot x^m = x^{n+m}$, which gives you $6x^{16}$ as the final result.

Try it: Calculate $(2 \times 10^7) \cdot (3 \times 10^9)$.

4 When adding and subtracting using scientific notation, you need to be a bit more careful. We need to have like terms in order for us to combine them. For example, $6x^8 + 2x^7$ cannot be simplified. In order to get around this, we need to rewrite the numbers so that they have the same power of 10.

In order to do that, we need to think about other representations of numbers. Here is a collection of expressions that are all equivalent to 5000:

$$5000 = 5000 \times 10^0 = 500 \times 10^1 = 50 \times 10^2 = 5 \times 10^3 = 0.5 \times 10^4 = 0.05 \times 10^5 = \cdots$$

The important feature to notice is that as the power of 10 increases, the other number decreases. This balancing act makes sense if you think about the overall result. If the overall result is not going to change, then if one number of the product gets larger, the other must get smaller.

Technically, these representations are not scientific notation. But it doesn't actually matter what your intermediate steps are as long as your final answer is in scientific notation.

Once that pattern is recognized, the calculations are simply a matter of execution. Here are two common approaches to the same calculation.

$$\begin{aligned}(6 \times 10^8) + (2 \times 10^7) &= (60 \times 10^7) + (2 \times 10^7)\\ &= 62 \times 10^7\\ &= 6.2 \times 10^8\end{aligned}$$

And here is the second:

$$\begin{aligned}(6 \times 10^8) + (2 \times 10^7) &= (6 \times 10^8) + (0.2 \times 10^8)\\ &= 6.2 \times 10^8\end{aligned}$$

Some prefer this first way because it avoids decimals. Some prefer the second way because it's shorter. It doesn't matter which way you do it as long as your steps are valid.

Try it: Calculate $(8 \times 10^6) + (6 \times 10^8)$. Give your final answer in scientific notation.

5 The same idea can be applied when the exponent is negative, but you need to be careful. As the exponent gets more negative, the power of 10 becomes smaller and so the other number must get bigger.

$$0.007 = 0.007 \times 10^0 = 0.07 \times 10^{-1} = 0.7 \times 10^{-2} = 7 \times 10^{-3} = 70 \times 10^{-4} = \cdots$$

Try it: Calculate $(5 \times 10^{-6}) + (4 \times 10^{-5})$. Give your final answer in scientific notation.

34.1 Scientific Notation - Worksheet 1

1

Convert the following list of numbers into scientific notation.

$38000000000 =$ $\quad$ $17500000000000000000 =$

$860000000000000 =$ $\quad$ $20000 =$

$3727000000000 =$ $\quad$ $6811000000000000 =$

There's no "work" to show here. Count carefully!

2

Convert the following list of numbers into scientific notation.

$0.00000000032 =$ $\quad$ $0.0000751 =$

$0.0000003 =$ $\quad$ $0.000000000866 =$

$0.00073 =$ $\quad$ $0.000000045 =$

3

Convert the following list of numbers into standard form.

$2.45 \times 10^7 =$ $\quad$ $8.12 \times 10^3 =$

$3.8 \times 10^5 =$ $\quad$ $5.318 \times 10^{10} =$

$7.5 \times 10^6 =$ $\quad$ $6.38 \times 10^9 =$

4

Convert the following list of numbers into standard form.

$1.86 \times 10^{-3} =$ $\quad$ $3.8 \times 10^{-6} =$

$6.83 \times 10^{-5} =$ $\quad$ $5 \times 10^{-9} =$

$8.931 \times 10^{-7} =$ $\quad$ $8.12 \times 10^{-2} =$

34.2 Scientific Notation - Worksheet 2

1

Convert the following list of numbers into scientific notation.

$680000000 =$

$110000000000 =$

$0.0000000138 =$

$1830000000 =$

$0.000008861 =$

$0.0000000000003 =$

2

Convert the following list of numbers into standard form.

$3.22 \times 10^{5} =$

$4.9 \times 10^{-6} =$

$7.69 \times 10^{-4} =$

$9.08 \times 10^{8} =$

$3.83 \times 10^{7} =$

$8.18 \times 10^{-5} =$

3

Convert the following numbers into both scientific notation and standard form.

	Scientific Notation		Standard Form
$23.86 \times 10^{-5} =$		$=$	
$480000 \times 10^{3} =$		$=$	
$0.000831 \times 10^{6} =$		$=$	

4

Determine the value of a that will make each of these equalities valid.

$5.383 \times 10^{6} = a \times 10^{5},\ a =$

$5.383 \times 10^{6} = a \times 10^{7},\ a =$

$5.383 \times 10^{6} = a \times 10^{4},\ a =$

$5.383 \times 10^{6} = a \times 10^{8},\ a =$

34.3 Scientific Notation - Worksheet 3

1 Calculate $(3 \times 10^5) + (6 \times 10^5)$ and $(8 \times 10^{-8}) + (9 \times 10^{-8})$. Give your final answer in scientific notation.

2 Calculate $(4 \times 10^7) + (7 \times 10^8)$ and $(6 \times 10^{-5}) + (9 \times 10^{-4})$. Give your final answer in scientific notation.

3 Calculate $(3 \times 10^4) \cdot (6 \times 10^5)$ and $(5 \times 10^{-8}) \cdot (8 \times 10^{-6})$. Give your final answer in scientific notation.

4 Calculate $(8 \times 10^4) \cdot (4 \times 10^{-6})$ and $(9 \times 10^{-3}) \cdot (2 \times 10^4)$. Give your final answer in scientific notation.

34.4 Scientific Notation - Worksheet 4

1 Calculate $(5 \times 10^5) + (8 \times 10^7)$ and $(8 \times 10^{-5}) - (5 \times 10^{-7})$. Give your final answer in scientific notation.

2 Calculate $(8 \times 10^{-3}) + (7 \times 10^{-2})$ and $(8 \times 10^{12}) - (5 \times 10^{11})$. Give your final answer in scientific notation.

3 Calculate $(7 \times 10^5) \cdot (8 \times 10^{-3})$ and $(2 \times 10^{-5}) \cdot (5 \times 10^4)$. Give your final answer in scientific notation.

4 Calculate $\frac{8 \times 10^4}{4 \times 10^{-6}}$ and $\frac{9 \times 10^{-5}}{3 \times 10^{-2}}$. Give your final answer in scientific notation.

Neither of these calculations should require a calculator.

34.5 Scientific Notation - Worksheet 5

In practice, calculations done in scientific notation are done by calculator. More advanced calculators can handle the scientific notation on its own, but others can't. We are going to practice calculations as if the calculators cannot handle scientific notation. The reason for this is to emphasize the logic of scientific notation.

Different calculators present scientific notation differently. You should learn how your specific calculator works.

The process of performing the calculation is the same as in the previous problems, except that instead of using mental arithmetic to calculate things like $40+8$ and $3 \cdot 5$, you would use the calculator to calculate things like $31.57 + 1.79$ and $(2.83) \cdot (8.9)$.

1 Calculate $(2.8619 \times 10^6) + (3.18 \times 10^4)$ using a calculator. Give your final answer in scientific notation.

2 Calculate $(1.86 \times 10^6) \cdot (4.03 \times 10^{-3})$ using a calculator. Give your final answer in scientific notation.

3 Calculate $\frac{8.136 \times 10^{-5}}{2.9 \times 10^4}$ using a calculator. Give your final answer in scientific notation.

34.6 Deliberate Practice: Scientific Notation

Focus on these skills:

- Carefully count out the zeros or decimal places in your conversion.
- Present your work legibly.

Instructions: Convert the number to both standard form and scientific notation.

1. 1300×10^{-2}
2. 0.78×10^{-2}
3. 0.053×10^{8}
4. 17000×10^{3}
5. 12.87×10^{-4}
6. 15.98×10^{7}
7. 0.00017×10^{-3}
8. 0.00086×10^{6}
9. 250000×10^{-8}
10. 39000×10^{4}

34.7 Closing Ideas

There was once a time when teachers would tell students, "You're going to need to learn these things because you won't always have a calculator in your pocket." It turns out that those teachers were wrong. Almost everyone these days has a cell phone on them most of the time, and so they almost always have access to a calculator. But the existence of those calculators has not changed the fact that people still need to develop mathematical reasoning.

As we saw in this section, there are sometimes problems that calculators can't to perform. And even if you have a calculator that does scientific notation, that doesn't mean you can actually use that functionality. So there are still cognitive gaps that need to be covered in one way or another. An unhealthy approach is simply to say "push these buttons to do scientific notation" and turn math back into another rule-based structure. The challenge is that for the majority of students, you will only see scientific notation in science classes and then only a few other times in the rest of their lives.

If we isolate scientific notation to a set of rules that will only be used in a few circumstances, then the most likely outcome is that students won't remember it at all. But if we connect it back to a conceptual idea (combining like terms), there is a greater chance that student will not only remember, but also have a better chance of re-understanding it if they forget. And this is an important benefit to thinking about math as a body of connected ideas instead of individual skills. It creates opportunities for students be able to use their experience and reasoning to rebuild lost knowledge, instead of simply being stuck at "I don't remember this."

34.8 Solutions to the "Try It" Examples

1

$$1.234 \times 10^5 = 123400 \qquad 9.87 \times 10^{-6} = 0.00000987$$

2

$$3480000000 = 3.48 \times 10^9 \qquad 0.0000736 = 7.36 \times 10^{-5}$$

3

$$(2 \times 10^7) \cdot (3 \times 10^9) = 6 \times 10^{16}$$

4

$$\begin{aligned}(8 \times 10^6) + (6 \times 10^8) &= (8 \times 10^6) + (600 \times 10^6)\\ &= 608 \times 10^6\\ &= 6.08 \times 10^8\end{aligned}$$

5

$$\begin{aligned}(5 \times 10^{-6}) + (4 \times 10^{-5}) &= (5 \times 10^{-6}) + (40 \times 10^{-6})\\ &= 45 \times 10^{-6}\\ &= 4.5 \times 10^{-5}\end{aligned}$$

Say Goodbye to the Mars Climate Orbiter: Unit Conversions

Learning Objectives:

- ☐ Convert mathematical relationships into conversion factors.
- ☐ Use conversion factors to perform unit conversions.
- ☐ Use scientific prefixes to create mathematical relationships.

The $300, 000, 000 NASA project called the *Mars Climate Orbiter* launched on December 11, 1998. Approximately 9 months later, it crashed into the surface of Mars and was declared a total loss. The culprit was a piece of hardware that gave distance in meters to a piece of software that was working in feet.

Search online for more information about the *Mars Climate Orbiter*.

This incident stands out among the many unit conversion errors (or simply the failure to convert units at all) because of a combination of the magnitude of the project and the high capacity minds that were working on it. Most errors of this type are not nearly as catastrophic, but that does not negate the importance of understanding how to perform unit conversions.

Units are names of measurements of quantities. Here are some examples:

These lists are not comprehensive.

- Time is measured in seconds, minutes, hours, days, and years.
- Lengths are measured in inches, feet, miles, centimeters, meters, and kilometers.
- Volumes are measured in tablespoons, cups, quarts, gallons, liters, and cubic meters.
- Quantities are measured in dozens, hundreds, thousands, millions, and moles.

No, moles is not a typo.

1 The most important feature about these units is that they represent specific quantities that can be related to each other. For example, there are 12 inches in a foot. And there are 60 seconds in a minute. These create mathematical relationships by simply translating the words into a formula:

$$\text{12 inches in a foot} \longrightarrow \text{1 foot} = \text{12 inches}$$
$$\text{60 seconds in a minute} \longrightarrow \text{60 seconds} = \text{1 minute}$$

These equations can be turned into fractions that we call conversion factors. Conversion factors have the property that they are equal to the number 1. Notice that every equality of this type creates two conversion factors.

$$\text{1 foot} = \text{12 inches} \longrightarrow \frac{\text{1 foot}}{\text{12 inches}} = 1 \quad \text{and} \quad 1 = \frac{\text{12 inches}}{\text{1 foot}}$$
$$\text{60 seconds} = \text{1 minute} \longrightarrow \frac{\text{60 seconds}}{\text{1 minute}} = 1 \quad \text{and} \quad 1 = \frac{\text{1 minute}}{\text{60 seconds}}$$

Try it: Write an equation that relates feet to yards, then use it to determine two conversion factors.

2 The importance of conversion factors being equal to the number 1 is that multiplying by 1 does not change the value of a number. With an appropriate choice of conversion factors, it is

For every value of x, $x \cdot 1 = x$.

possible to make original unit cancel out, leaving you with the value in the new unit. Here is an example:

$$\begin{aligned} 8 \text{ minutes} &= 8 \text{ minutes} \cdot 1 && \text{Multiplying by 1} \\ &= 8 \text{ minutes} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} && \text{Conversion factor} \\ &= 8 \cancel{\text{minutes}} \cdot \frac{60 \text{ seconds}}{1 \cancel{\text{minute}}} && \text{Cancel the units} \\ &= 480 \text{ seconds} \end{aligned}$$

This example was drawn out with details for emphasis. Your presentation can be shortened.

$$\begin{aligned} 8 \text{ minutes} &= 8 \cancel{\text{minutes}} \cdot \frac{60 \text{ seconds}}{1 \cancel{\text{minute}}} && \text{Cancel the units} \\ &= 480 \text{ seconds} \end{aligned}$$

Try it: Convert 36 feet to yards using a conversion factor.

If you choose the wrong conversion factor, the units won't cancel out.

3

You may be aware that there are certain prefixes that apply the the unit. These prefixes are another way that scientists avoid having to write really long numbers. Here are some of the common prefixes and their meaning:

Prefix	Symbol	Value as Power of 10	Value in Standard Form
nano	n	10^{-9}	0.000000001
micro	μ	10^{-6}	0.000001
milli	m	10^{-3}	0.001
centi	c	10^{-2}	0.01
deci	d	10^{-1}	0.1
deka	da	10^{1}	10
hecto	h	10^{2}	100
kilo	k	10^{3}	1000
mega	M	10^{6}	1000000
giga	G	10^{9}	1000000000
tera	T	10^{12}	1000000000000

μ is the Greek letter "mu."

These prefixes are placed in front of a unit to change its value. For example, a kilometer is 1000 meters and a millimeter is 0.001 meters. This gives us mathematical relationships that we can use to write conversion factors. While it is not wrong to use decimals in the conversion factors with these prefixes, it's generally considered better to use integers. So instead of using $\frac{0.001 \text{ meters}}{1\text{millimeter}}$, we would usually use $\frac{1 \text{ meter}}{1000 \text{ millimeters}}$.

Try it: Write an equation that relates kilograms to grams, then use it to determine two conversion factors.

35.1 Unit Conversions - Worksheet 1

1 Write an equation that relates inches to centimeters, then use it to determine two conversion factors.

You may not know all of these conversions. You can look them up on the internet if you're not sure.

2 Convert 12 inches into centimeters.

3 Convert 100 centimeters into inches.

4 Write an equation that relates cups to quarts, then use it to determine two conversion factors.

5 Convert 32 cups into quarts.

6 Convert 4 quarts into cups.

35.2 Unit Conversions - Worksheet 2

1 Write an equation that relates inches to kilowatts to watts, then use it to determine two conversion factors.

2 Write an equation that relates inches to deciliters to liters, then use it to determine two conversion factors.

3 Write an equation that relates nanometers to meters, then use it to determine two conversion factors.

4 Write an equation that relates hectares to ares, then use it to determine two conversion factors.

A hectare is the area of a square with side lengths of 100 meters. However, it is not equal to 100 square meters. We'll explore this in a later worksheet.

5 Write an equation that relates centimeters to meters and another equation that relates meters to kilometers. Use these two equations together to determine two conversion factors that relate kilometers to centimeters.

35.3 Unit Conversions - Worksheet 3

In science courses, you will encounter some interesting units that are built to describe specific situations. Although you may not have any intuition with these, as long as you have a formula and you understand the method, you can be begin to work with the problems.

1 A mole of objects is 6.022×10^{23} of those objects. For example, a mole of carbon atoms is 6.022×10^{23} carbon atoms. How many molecules of oxygen are in 5.7 moles of oxygen?

2 An astronomical unit is approximately 1.496×10^{11} meters. This is the approximate distance from the earth to the sun. Mars is approximately 1.52 astronomical units from the sun. About how far is it from the sun to Mars in meters?

3 Words such as millions and billions can also be used as a unit conversion. This is often used when talking about finances at a state or national level. Earlier, we talked about the \$300000000 Mars Climate Orbiter. This could have been written as \$300 million.

Write the quantity 2753.78 billion in standard form and using scientific notation.

1 thousand	$= 10^3$
1 million	$= 10^6$
1 billion	$= 10^9$
1 trillion	$= 10^{12}$

4 A light-year is the distance that light can travels in one year in a vacuum. This distance is approximately 9.46×10^{12} kilometers. The distance between the Milky Way galaxy and the Andromeda galaxy is approximately 2.5 million light-years. Approximately how many kilometers is it between the two galaxies?

35.4 Unit Conversions - Worksheet 4

1 A stone is a measure of weight that is commonly used in the UK and Ireland. One stone is equal to 14 pounds. If an object weighs 37 stone, how many pounds does it weigh?

2 A smoot is a unit of measurement devised as a prank by a fraternity at MIT. It is considered to be 67 inches, which corresponds to the height of Oliver Smoot, who was used to measure Harvard bridge. The bridge was determined to have a length of approximately 364.4 smoots. Convert this distance to feet.

Convert smoots to inches, then inches to feet.

3 When thinking about very large or very small quantities, it is sometimes useful to relate them to quantities that you are more familiar with. For example, we can think of 1 home $= \$400000$ as a relationship for converting dollars into homes. What is the equivalent of $1 billion in homes?

Go through the process of creating a conversion factor.

4 Transistors are an important component for modern electronics. The smallest transistors are about 7 nanometers in size. A human hair is approximately 100 micrometers in diameter. How many transistors would need to be lined up to equate to the thickness of a human hair?

35.5 Unit Conversions - Worksheet 5

A compound unit is a unit that mixes multiple other units together. Some common compound units are speeds like miles per hour ($\frac{\text{miles}}{\text{hours}}$) and pressures like pounds per square inch ($\frac{\text{pounds}}{\text{inch}^2} = \frac{\text{pounds}}{\text{inch}\cdot\text{inch}}$). When converting these, every unit much be converted individually. Here is an example of converting 10 feet per day into inches per week:

$$\begin{aligned} 10\frac{\text{feet}}{\text{day}} &= 10\frac{\cancel{\text{feet}}}{\cancel{\text{day}}} \cdot \frac{12 \text{ inches}}{1 \cancel{\text{ foot}}} \cdot \frac{7 \cancel{\text{ days}}}{1 \text{ week}} \\ &= 840\frac{\text{inches}}{\text{week}} \end{aligned}$$

1 One mile is equal to about 1.6 kilometers. If a car is traveling 75 miles per hour, how many kilometers per minute is it traveling?

2 Water has a density of about 8.34 pounds per gallon. Convert this to ounces (weight) per ounce (liquid).

3 A hectare is the area of a square that is 100 meters on each side. Determine the conversion factor for hectares to square meters.

35.6 Deliberate Practice: Unit Conversions

Focus on these skills:

- You may need to use the internet to determine the appropriate conversion factor.
- Explicitly write out the conversion factor and show the cancellation step.
- Present your work legibly.

Instructions: Convert the number to both standard form and scientific notation.

1. Convert 39 deciliters to milliliters.
2. Convert 4 kilometers to centimeters.
3. Convert 13 micrograms to milligrams.
4. Convert 3 astronomical units to meters.
5. Convert 8 nautical miles to feet.
6. How many ounces are in 12 cups?
7. How many pints are in 3 gallons?
8. How many teaspoons are in 9 tablespoons?
9. How many ounces are in 3 pounds?
10. How many liters are in 2 gallons?

35.7 Closing Ideas

Understanding how to manipulate units is an important skill for science courses, but these types of conversions happen all the time in ways that you might not expect. For example, if you have $10 and a plate of 3 street tacos costs $2.50, you can convert $10 to 4 plates of tacos, and 4 plates of tacos to 12 individual street tacos. You probably won't need to write out the conversion factors, and that's okay. Remember that the goal is to understand what unit conversions do and how they work. The goal is not that you would learn that math is a set of rules you must follow.

This example may seem intuitive, but that intuition comes with familiarity and experience. Even though you may not have found a 3 street tacos for $2.50 deal, you understand how it works. If you have an intuitive sense for how these conversions work and you understand how unit manipulations work, then it's not as difficult to solve problems in situations where you have less familiarity.

Every now and then, there's an internet math meme that shows up that is an example of unit conversions gone wrong. Here is one example:

> The winner of the next lottery will win $1300 million. There are 300 million people in the US.
>
> $$\frac{\$1300 \text{ million}}{300 \text{ million}} = \$4.33 \text{ million}$$
>
> Why not just give everyone $4.33 million?

Can you spot the error? (Hint: Think about we have manipulated units in fractions throughout this entire section.)

No, we're not going to give you the answer.

35.8 Solutions to the "Try It" Examples

1

$$3 \text{ feet} = 1 \text{ yard} \quad \longrightarrow \quad \frac{3 \text{ feet}}{1 \text{ yard}} = 1 \quad \text{and} \quad 1 = \frac{1 \text{ yard}}{3 \text{ feet}}$$

2

$$\begin{aligned} 36 \text{ feet} &= 36 \cancel{\text{ feet}} \cdot \frac{1 \text{ yard}}{3 \cancel{\text{ feet}}} && \text{Cancel the units} \\ &= 12 \text{ yards} \end{aligned}$$

3

$$1000 \text{ grams} = 1 \text{ kilogram} \quad \longrightarrow \quad \frac{1000 \text{ grams}}{1 \text{ kilogram}} = 1 \quad \text{and} \quad 1 = \frac{1 \text{ kilogram}}{1000 \text{ grams}}$$

Final Reflection Reflection

Congratulations! You are at the end of this book. As you have worked your way through these materials, we hope that you've started to really embrace the idea that thinking mathematically is both important and useful. We've covered a lot of material, and even though many of the concepts were simple, that doesn't mean that this was an easy journey.

And now that we are at the very end, we're going to look back over the scope of topics that we covered.

The main trunk of this book focused on making sure that you had an understanding of the core algebraic manipulations that you will need to be successful in a college level math course:

- Basic Algebraic Presentation
- Variables in Expressions and Equations
- Like and Unlike Terms
- Simplifying Expressions and Solving Equations
- Variables and Substitutions
- The Properties of Exponents
- Products of Polynomials
- Common Factors
- Factoring Quadratic Polynomials
- Reading Mathematical Expressions

The first branch covered linear equations and coordinate plane:

- Lines and the Coordinate Plane
- Slope-intercept form
- Solving systems of equations by substitution
- Solving systems of equations by elimination

The second branch covered fractions and decimals:

- Fraction basics
- Fraction Addition and Subtraction
- Fraction Multiplication
- Fraction division
- Decimal addition and subtraction
- Multiplying Decimals and Percents

The third branch covered the underlying concepts that we use in arithmetic:

- The Number Line and Base-10 blocks as Visualizations of the Integers
- Visualizations of Addition and the Addition Algorithm
- Visualizations of Subtraction and the Subtraction Algorithm
- Integer Chips as a Representation of Negative Numbers in Addition and Subtraction Calculations
- Movement on the Number Line for Negative Numbers in Addition and Subtraction Calculations
- Visualizations of Multiplication as A Groups of B and Area
- Visualizations of Division as Making Groupings and Equal Distribution

The fourth branch covered a few key application ideas:

- General Word Problems and Avoiding the Trap of Key Words
- Scientific Notation
- Unit Conversions

If you worked your way through all of these topics, you should feel very confident that you have the basic foundation you need to be successful at your introductory college level mathematics course. You may not consider yourself a “math person” (yet), but you are well on your way for having completed this book.

Questions About the Content

1 Were there any topics that you had seen before, but you understand better as a result of working through it again?

2 Were there any ideas that you had never seen before?

3 Based on your experience, which of these ideas seems the most important to understand well?

4 Did any part of the presentation make you curious about math in a way that went beyond the material? Are there questions or ideas that you would like to explore?

Examples:

- What other key word traps are there?
- What are other applications of unit conversions?

Questions About You

1 How has your mathematical writing continued to evolve from the previous pause for reflection? Do you find yourself thinking in different ways?

2 What was your biggest "Aha!" moment of the book?

3 What is the biggest mathematical connection that you made?

4 How is your mathematical confidence coming out of these book? Do you feel more or less confident in your mathematical knowledge than before the book? Explain why you think you feel the way you do.